Introduction to Dynamic Programming

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Overview

This slide is to introduce three representations of the infinite horizon problem using **Neoclassical Growth Model** as example.

We will go over the **Solow model** first to get the sense of problem we are working with.

The three representations are:

1. Date 0 economy
2. Sequential formulation
3. Recursive formulation

Solow Model

- Infinite horizon: $t = 0, 1, 2, \ldots$
- Single homogeneous good produced each period.
- Output Y_t can be used for consumption C_t or investment I_t.
  - the share s is constant: $I_t = sY_t$
- Labor force is constant over time, i.e., $L_t = L, \forall t$.
- Capital K_t depreciates at rate $\delta \in (0, 1)$.
  - Capital law of motion: $K_{t+1} = (1 - \delta)K_t + I_t$.
- Production function: $Y_t = F(K_t, L_t)$

Analysis: Solow Model

- Since saving rate s is constant $\Rightarrow C_t = (1-s)Y_t \Rightarrow u(C_t)$ predetermined, no need for consumer problem.

- Three equations:

$$C_t + I_t = F(K_t, L) \tag{1}$$

$$K_{t+1} = (1-\delta)K_t + I_t \tag{2}$$

$$I_t = sF(K_t, L) \tag{3}$$

can substitute into one:

$$K_{t+1} = g(K_t) \equiv (1-\delta)K_t + sF(K_t, L), \tag{4}$$

- Given K_t, K_{t+1} is determined at time $t \Rightarrow$ capital is a **state** variable.

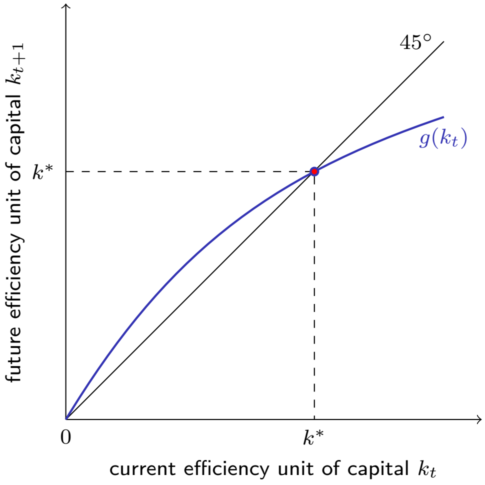
Properties: Solow Model

- Need some assumptions to hold, and **Cobb-Douglas** matches all
  - $F(K_t, L_t) = AK_t^{\alpha} L_t^{1-\alpha}$
- Exists a steady state such that $K_t = K_{t+1} = K^*$, i.e.,

$$K^* = g(K^*) \equiv (1 - \delta)K^* + sF(K^*, L), \tag{5}$$

- and the nontrivial solution is also unique (figure next slide)

Properties: Solow Model (Cont.)



Neoclassical Growth Model: Set up

- Difficulties with Solow Model: exogenous saving rate.
  - **how** arrived at s? Is s **optimal**?

- Micro-foundation: rep. consumer makes consumption-saving decision.

- No externalities, and thus can solve in Social planner's problem.

- Assume rep. consumer lives for ∞ period with **additive** separability:

$$U(C_0, C_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(C_t), \tag{6}$$

where function $u(\cdot)$ is the same for every period, and β is subjective discount factor.

Neoclassical Growth Model: Set up (Cont.)

- Assumes no labor (for the sake of sanity)
- Two goods are trading:
  - firm $\to$ consumer: consumption goods (c_t) with price p_t
  - consumer $\to$ firm: capital accumulation (k_t) with price r_t

Date 0 Representation

A Date 0 C.E. is **prices** $\{p_t, r_t\}_{t=0}^{\infty}$ and **quantities** $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ such that

1. $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves household's problem,

$$\max_{\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{7}$$

$$\text{subject to} \quad c_t \geq 0, \forall t = 0, 1, \ldots \tag{8}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + (1-\delta)k_t), \forall t \tag{9}$$

2. $\{k_{t+1}^*\}_{t=0}^{\infty}$ solves firm's problem at each $t = 0, 1, \ldots$

$$\max_{k_t} p_t f(k_t) - p_t r_t k_t \tag{10}$$

3. Goods market clear: $c_t^* + k_{t+1}^* = f(k_t^*) + (1-\delta)k_t^*$

Discussion on Date 0 Representation

- p_t is the relative price of c_t **in units of** $c_0 \Rightarrow p_0 = 1$.

- $p_t r_t$ is the relative price of capital **in units of** c_0

- Firm's problem is static, implies $r_t = D_k f(k_t)$

- Use **LaGrange multiplier** λ, we derive the FOC for c_t and k_{t+1} are

$$[c_t]: \quad \beta^t u'(c_t) = \lambda p_t$$
$$[k_{t+1}]: \quad p_t = p_{t+1}(r_{t+1} + 1 - \delta)$$

- If we divide both p_t and p_{t+1}, we get **Euler equation**:

$$\frac{p_t}{p_{t+1}} = \frac{u'(c_t)}{\beta u'(c_{t+1})} = (r_{t+1} + 1 - \delta) \Rightarrow u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta)$$

Sequential Representation

A sequential C.E. is **prices** $\{r_t\}_{t=0}^{\infty}$ and **quantities** $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ such that

1. $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves household's problem,

$$\max_{\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{11}$$

$$\text{subject to} \quad c_t \geq 0, \forall t = 0, 1, \ldots \tag{12}$$

$$c_t + k_{t+1} \leq r_t k_t + (1-\delta)k_t, \forall t = 0, 1, \ldots \tag{13}$$

$$\lim_{t\to\infty} \left(\prod_{s=1}^{t} (r_t + 1 - \delta) \right)^{-1} k_{t+1} = 0 \tag{14}$$

2. $\{k_{t+1}^*\}_{t=0}^{\infty}$ solves firm's problem at each $t = 0, 1, \ldots$

$$\max_{k_t} f(k_t) - r_t k_t \tag{15}$$

3. Goods market clear: $c_t^* + k_{t+1}^* = f(k_t^*) + (1-\delta)k_t^*$

Discussion on Sequential Representation

- Here we have budget constraint at every possible t, rather than one.
- Need **LaGrange multiplier** λ_t for each budget constraint!
- FOC for c_t and k_{t+1} are

$$[c_t]: \quad \beta^t u'(c_t) = \beta^t \lambda_t \Rightarrow u'(c_t) = \beta \lambda_t$$
$$[k_{t+1}]: \quad \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}(r_{t+1} + 1 - \delta) \Rightarrow \lambda_t = \beta \lambda_{t+1}(r_{t+1} + 1 - \delta)$$

- and still, we can the same **Euler equation**:

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta)$$

- Equation (14) is the transversality condition: avoid Ponzi scheme

Intro: Recursive Representation

- In the sequential representation, at each date t, household is solving **exactly the same** utility optimization problem, so we can write it as:

$$\max_{c_t, k_{t+1}} u(c_t) + \overbrace{\sum_{s=t+1}^{\infty} \beta^s u(c_s)}^{\text{not related to } c_t} \tag{16}$$

$$\text{subject to} \quad c_t + k_{t+1} \leq r_t k_t + (1-\delta) k_t \tag{17}$$

$$c_{t+1} + k_{t+2} \leq r_{t+1} k_{t+1} + (1-\delta) k_{t+1} \tag{18}$$

- Observing this, instead of finding the **level** of the prices and quantities, we find the **function** of prices and quantities that express the same problem that household is solving **at each** t.

- Note that HH cannot change prices, and thus prices depends on the **aggregate** state variable, i.e., aggregate capital $\bar{K}$. In equilibrium $\bar{K} = k$.

Recursive Representation

A recursive C.E. is a set of functions for **prices** $\{r(\bar{K})\}$ and **quantities** $\{G(\bar{K}), g(k, \bar{K})\}$ and value $V(k, \bar{K})$ such that

1. $V(k, \bar{K})$ solves household's problem,

$$V(k, \bar{K}) = \max_{c, k' \geq 0} \left(u(c) + \beta V(k', \bar{K}')\right) \tag{19}$$

$$\text{subject to} \quad c + k' = (r(\bar{K}) + 1 - \delta)k \tag{20}$$

$$\bar{K}' = G(\bar{K}) \tag{21}$$

2. Prices are competitively determined, i.e., firm's problem implies

$$r(\bar{K}) = f'(\bar{k}),$$

3. Individual decisions are consistent with aggregates when $k = \bar{K}$, i.e.,

$$G(\bar{K}) = g(\bar{K}, \bar{K})$$

Discussion on Recursive Representation

- Why?! $\because$ The only formulation we can put it on computer!
- Date 0: how could you code the budget constraint w/ infinite sum?
- Sequential: infinite number of budget constraint...
- Recursive: through recursion, we can keep iterate on same problem until it **converges** to a fixed point.
- Difficulties: for each C.E., need to identify the **structure** of the question such that we can represent that structure using individual and aggregate state variables.