there's capital required in the production function

$Y = (K^d)^a (N^d)^{1-a}, a \in [0,1]$

Firm is renting capital from the consumer.

consumers are endowed with $K^s = \hat{k}$

utility function $U(C,l) = \frac{C^{1-b}}{1-b} + \frac{l^{1-d}}{1-d}$

1. Formulate the competitive equilibrium

a. consumer maximize utility function subject to budget constraint

$$\max_{C,l} U(C,l) = \frac{C^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \tag{1}$$

subject to

$$C \leqslant \underbrace{w \overbrace{(h-l)}^{\text{labor supply } N^s}}_{\text{labor income}} + \underbrace{r K^s}_{\text{capital income}} + \underbrace{\pi}_{\text{firm's profit}} - \underbrace{T}_{\text{lump-sum tax}} \tag{2}$$

b. firm maximize profit

$$\max_{K^d, N^d} \pi = (K^d)^a (N^d)^{1-a} - w N^d - r K^d \tag{3}$$

$$\frac{\partial \pi}{\partial N^d} = (1-a)\,(K^d)^a\,(N^d)^{-a} - w = 0 \Rightarrow w = (1-a)\,(K^d)^a\,(N^d)^{-a} \tag{4}$$

$$\frac{\partial \pi}{\partial K^d} = a\,(K^d)^{a-1}\,(N^d)^{1-a} - r = 0 \Rightarrow r = a\,(K^d)^{a-1}\,(N^d)^{1-a} \tag{5}$$

c. government collect the taxes to satisfy the exogenous government spending.

$$T^* = G \tag{6}$$

d. labor market clears, equilibrium wage w^* such that

$$N^d = N^s \tag{7}$$

e. capital market clears, equilibrium rental rate r^* such that

$$K^d = K^s \tag{8}$$

2. Social planner's problem

$$\begin{aligned} & \max_{C,l,N,Y,K} U(C,l) = \frac{C^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \\ s.t. \quad & C + G = Y \\ & Y = K^a N^{1-a} \\ & N = h - l = 1 - l \\ & K = \hat{k} \end{aligned}$$

$$Y = \hat{k}^a\,(1-l)^{1-a} \tag{9}$$

$$C = Y - G = \hat{k}^a\,(1-l)^{1-a} - G \tag{10}$$

$$\max_{l} U(C(l), l) = \frac{(\hat{k}^a\,(1-l)^{1-a} - G)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \tag{11}$$

Derive first order condition of equation 9

Derive the derivative U with respect to l

$\frac{\partial \frac{l^{1-d}}{1-d}}{\partial l} = \frac{\partial \frac{1}{1-d} \times l^{1-d}}{\partial l} = l^{-d}$

$\frac{\partial U(C(l),l)}{\partial C} = \frac{C^{1-b}}{1-b} = C^{-b} = (\hat{k}^a (1-l)^{1-a} - G)^{-b}$

$\frac{\partial (\hat{k}^a (1-l)^{1-a})}{\partial l} = \hat{k}^a \times (1-a) \times (1-l)^{-a} \times (-1)$

$$\begin{aligned}
\frac{dU(C(l),l)}{dl} &= \frac{\partial U(C(l),l)}{\partial C} \times \frac{\partial C(l)}{\partial l} + \frac{\partial U(C(l),l)}{\partial l} = 0 \\
&= \underbrace{(\hat{k}^a (1-l)^{1-a} - G)^{-b}}_{\frac{\partial U(C(l),l)}{\partial C}} \times \underbrace{\hat{k}^a \times (1-a) \times (1-l)^{-a} \times (-1)}_{\frac{\partial C(l)}{\partial l}} + \underbrace{l^{-d}}_{\frac{\partial U(C(l),l)}{\partial l}} = 0 \\
&= (\hat{k}^a (1-l)^{1-a} - G)^{-b} \times \hat{k}^a \times (1-a) \times (1-l)^{-a} = l^{-d}
\end{aligned}$$