Asset Pricing in Endowment Economy

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Overview

How to value an asset?

Lucas (1978) answer this question by the "fruit tree" story:

- each representative household initially owns one fruit tree,
- the fruits (dividend) from tree is uncertain each period, and
- household **cannot store** the fruits.
- To achieve intertemporal substitution, HH can exchange the **property right** of the tree ⇒ share of the tree.
 - Each tree is identical \Rightarrow the randomness of fruit is identical.
 - If fruits (dividend) vary over time, how do share price varies?
 - What is the fundenmental value of an asset?
 - How to compare fundenmental value between different assets?

Model Setting

Let

- z be the fruit/dividend per tree
- *s* be the share of a tree,

Household's problem:

$$V(s,z) = \max_{s',c} u(c) + \beta \mathbb{E}_{z'|z}[V(s',z')]$$
(1)
subject to $c + ps' \le (z+p)s$ (2)

Equilibrium outcome is trivial: since the only resource is z, so goods market clearing condition is c = z, which leads to s = 1 in equilibrium.

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What is share prices?

- Fundamental value of an asset is the expected discounted NPV of all its future payoff.
- In the case of firm share the payoff is the dividend.
- and we are going to derive the share price by solving the optimal choice of share.

Recall HH's problem at time $t\ {\rm can}\ {\rm be}\ {\rm written}\ {\rm as}$

$$V(s,z) = \max_{s',c} u(c) + \beta \mathbb{E}_{z'|z}[u(c')] + \beta^2 \mathbb{E}_{z'|z}[V(s'',z'')]$$
(3)

subject t

ct to
$$c + ps' \le (z + p)s$$
 (4)

$$c' + p's'' \le (z' + p')s'$$
 (5)

Optimal Choice of Share

Substitute,

$$\begin{split} V(s,z) &= \max_{s'} u \left((z+p)s - ps' \right) \\ &+ \beta \mathbb{E}_{z'|z} \left[u \left((z'+p')s' - p's'' \right) \right] \\ &+ \beta^2 \mathbb{E} [V(s'',z'')] \end{split}$$

FOC,

 $[s']: \quad u'(c)p = \beta \mathbb{E}_{z'|z} \left[u'(c')(z'+p') \right] = \beta \mathbb{E}_{z'|z} \left[u'(c')z' + u'(c')p' \right] \quad \textbf{(6)}$

Notice that the current price p depends on future price p'!

Fundamental Value of Share: Euler Equation Update

Nothing stops us to figure out what p' would looks like, and it should be update (6) one period forward:

$$u'(c')p' = \beta \mathbb{E}_{z''|z'} \left[u'(c'')(z'' + p'') \right]$$

So (6) will be

$$u'(c)p = \mathbb{E}_{z'|z} \left[\beta u'(c')z' + \beta^2 \mathbb{E}_{z''|z'} \left[u'(c'')(z'' + p'') \right] \right]$$
(7)
= $\mathbb{E}_{z'',z'|z} \left[\beta u'(c')z' + \beta^2 u'(c'')z'' + \beta^2 u'(c'')p'' \right]$ (8)

Repeat substitution and get

$$u'(c)p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) z_{t+j} \right] + \lim_{j \to \infty} \beta^j u'(c_{t+j}) p_{t+j}$$
(9)

Fundamental Value of Share

By transversality condition,

$$\lim_{j \to \infty} \beta^j u'(c_{t+j}) p_{t+j} = 0 \tag{10}$$

and thus,

$$u'(c)p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) z_{t+j} \right]$$
(11)

with goods market clearing condition $c_t = z_t$,

$$p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j} \right]$$
(12)

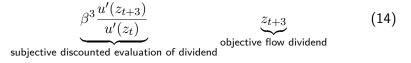
Fundamental Value of Share: Analysis

$$p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j} \right]$$
(13)

Risk Premium

Appendix

Take one of the period, say j = 3, the **evaluation** of dividend z_{t+3} is



Value of Share

Thus, we show that the share price p represents the expectation of subjective discounted evaluation of all future dividends.

Comparison between Evaluations

How to compare the value of different assets?

- Risk premium: measure of excess return that is required by an individual to compensate being subjected to an increased level of risk.
 - i.e., the percentage compensation required for household to take risk.

- Share is one kind of risky asset: the fruits from the tree is random.
- If household is able to access to risk-free asset, in equilibrium household should be indifferent between buying risk-free asset or risky share.

Access to Treasury Bill

Extension: HH can access both treasury bill and risky share.

- Buy share s at price p, get return (z' + p') next period
- Buy treasury bill a at price q, get return 1 next period

Budget constraint:

$$c + ps' + qa' \le [z + p]s + a \tag{15}$$

Following the same approach before (blackboard!), we know the FOCs are

$$[a']: \quad u'(c)q = \beta \mathbb{E}_{z'|z} \left[u'(c') \right] \Rightarrow q = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \right]$$
(16)

$$[s']: \quad u'(c)p = \beta \mathbb{E}_{z'|z} \left[u'(c')z' + u'(c')p' \right]$$
(17)

No-arbitrage Condition

Household are indifferent to treasury bill or risky share in equilibrium. Math representation can be observed by rearranging two FOCs:

$$[a']: \quad 1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times \frac{1}{q} \right]$$
(18)
$$[s']: \quad 1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times \frac{z' + p'}{p} \right]$$
(19)

Here, we derived expressions for the rate of return:

- Gross return on share: $e(z, z') \equiv \frac{z'+p'}{p}$
- Gross return on risk-free bond: $R(z) \equiv \frac{1}{q}$

Value of Share

Risk Premium

Appendix

Risk Premium: Derivation

From conditional covariance relation:

$$cov_{z}(A, B) = \mathbb{E}_{z'|z} \left[\left(A - \mathbb{E}_{z'|z}[A] \right) \left(B - \mathbb{E}_{z'|z}[B] \right) \right]$$
(20)
$$= \mathbb{E}_{z'|z} \left[AB \right] - \mathbb{E}_{z'|z}[A] \mathbb{E}_{z'|z}[B]$$
(21)

Equation (19) is $\mathbb{E}_{z'|z}[AB]$, where $A = rac{eta u'(c')}{u'(c)}$ and B = e(z,z'), and thus

$$1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times e(z, z') \right]$$

$$= cov_z \left[\beta \frac{u'(c')}{u'(c)}, e(z, z') \right]$$
(22)
(23)

+
$$\underbrace{\mathbb{E}_{z'|z}\left[\beta \frac{u'(c')}{u'(c)}\right]}_{\sum} \mathbb{E}_{z'|z}\left[e(z,z')\right]$$
 (24)

 $\equiv \frac{1}{R(z)}$, return on bond

Value of Share

Risk Premium: Derivation (Cont.)

Therefore

$$1 = cov_{z} \left[\frac{\beta u'(c')}{u'(c)}, e(z, z') \right]$$

$$+ \mathbb{E}_{z'|z}[e(z, z')] \times \frac{1}{R(z)}$$

$$\underbrace{\frac{\mathbb{E}_{z'|z}[e(z, z')] - R(z)}{R(z)}}_{\text{risk premium}} = -cov_{z} \left[\underbrace{\frac{\beta u'(c')}{u'(c)}}_{\text{SDF}}, e(z, z') \right]$$

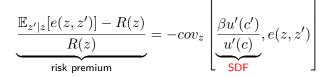
$$(25)$$

$$(26)$$

$$(27)$$

Stochastic discounting factor (SDF) / pricing kernel: HH's discounted percentage change in marginal utility.

Risk Premium: Discussion



1 If return on share, e(z, z'), is uncorrelated with consumption change $c \rightarrow c'$, then risk premium is 0

- HH not bearing risk \Rightarrow no need to compensate HH
- **2** If return on share, e(z, z'), is high when c' is low, i.e., e(z, z') and SDF are positively correlated, then risk premium is **negative**
 - HH is benefiting from buying this asset \Rightarrow need to charge HH!
- **③** If return on share, e(z, z'), is low when c' is low, i.e., e(z, z') and SDF are negatively correlated, then risk premium is **positive**
 - HH is bearing risk \Rightarrow need to compensate HH

References

Appendix

References

References I

Lucas, Robert E. (1978) "Asset Prices in an Exchange Economy," *Econometrica*, 46 (6), 1429, 10.2307/1913837.