Asset Pricing in Endowment Economy

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Overview

How to value an asset?

- Lucas (1978) answer this question by the "fruit tree" story:
  - each representative household initially owns **one** fruit tree,
  - the fruits (dividend) from tree is **uncertain** each period, and
  - household **cannot store** the fruits.
- To achieve intertemporal substitution, HH can exchange the **property right** of the tree $\Rightarrow$ share of the tree.
  - Each tree is identical $\Rightarrow$ the randomness of fruit is identical.
  - If fruits (dividend) vary over time, how do share price varies?
  - What is the fundenmental value of an asset?
  - How to compare fundenmental value between different assets?

Model Setting

Let

- z be the fruit/dividend per tree
- s be the share of a tree,

Household's problem:

$$V(s,z) = \max_{s',c} u(c) + \beta \mathbb{E}_{z'|z}[V(s',z')] \tag{1}$$

$$\text{subject to} \quad c + ps' \leq (z+p)s \tag{2}$$

Equilibrium outcome is trivial: since the only resource is z, so goods market clearing condition is $c = z$, which leads to $s = 1$ in equilibrium.

☺ The End ☺. . . ?

What is share prices?

- Fundamental value of an asset is the expected discounted NPV of all its future payoff.
- In the case of firm share the payoff is the dividend.
- and we are going to derive the share price by solving the **optimal choice of share**.

Recall HH's problem at time t can be written as

$$V(s,z) = \max_{s',c} u(c) + \beta \mathbb{E}_{z'|z}[u(c')] + \beta^2 \mathbb{E}_{z'|z}[V(s'',z'')] \quad (3)$$

$$\text{subject to} \quad c + ps' \leq (z+p)s \quad (4)$$

$$c' + p's'' \leq (z'+p')s' \quad (5)$$

Optimal Choice of Share

Substitute,

$$
\begin{aligned}
V(s,z) = \max_{s'} u\left((z+p)s - ps'\right) \\
+ \beta \mathbb{E}_{z'|z}\left[u\left((z'+p')s' - p's''\right)\right] \\
+ \beta^2 \mathbb{E}[V(s'', z'')]
\end{aligned}
$$

FOC,

$$
[s']: \quad u'(c)p = \beta \mathbb{E}_{z'|z}\left[u'(c')(z'+p')\right] = \beta \mathbb{E}_{z'|z}\left[u'(c')z' + u'(c')p'\right] \quad (6)
$$

Notice that the current price p depends on future price p'!

Fundamental Value of Share: Euler Equation Update

Nothing stops us to figure out what p' would looks like, and it should be update (6) one period forward:

$$u'(c')p' = \beta \mathbb{E}_{z''|z'} \left[u'(c'')(z'' + p'') \right]$$

So (6) will be

$$u'(c)p = \mathbb{E}_{z'|z} \left[\beta u'(c')z' + \beta^2 \mathbb{E}_{z''|z'} \left[u'(c'')(z'' + p'') \right] \right] \quad (7)$$

$$= \mathbb{E}_{z'',z'|z} \left[\beta u'(c')z' + \beta^2 u'(c'')z'' + \beta^2 u'(c'')p'' \right] \quad (8)$$

Repeat substitution and get

$$u'(c)p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) z_{t+j} \right] + \lim_{j \to \infty} \beta^j u'(c_{t+j}) p_{t+j} \quad (9)$$

Fundamental Value of Share

By transversality condition,

$$\lim_{j\to\infty} \beta^j u'(c_{t+j})p_{t+j} = 0 \tag{10}$$

and thus,

$$u'(c)p_t = \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j})z_{t+j}\right] \tag{11}$$

with goods market clearing condition $c_t = z_t$,

$$p_t = \mathbb{E}_t\left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j}\right] \tag{12}$$

Fundamental Value of Share: Analysis

$$p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j} \right] \tag{13}$$

Take one of the period, say $j = 3$, the **evaluation** of dividend z_{t+3} is

$$\underbrace{\beta^3 \frac{u'(z_{t+3})}{u'(z_t)}}_{\text{subjective discounted evaluation of dividend}} \qquad \underbrace{z_{t+3}}_{\text{objective flow dividend}} \tag{14}$$

Thus, we show that the share price p represents the expectation of subjective discounted evaluation of all future dividends.

Comparison between Evaluations

How to compare the value of different assets?

- **Risk premium**: measure of excess return that is required by an individual to compensate being subjected to an increased level of risk.
  - i.e., the percentage compensation required for household to take risk.
  - math: $\frac{\text{return on share} - \text{return on risk-free}}{\text{return on risk-free}}$
- Share is one kind of risky asset: the fruits from the tree is random.
- If household is able to access to risk-free asset, in equilibrium household should be **indifferent** between buying risk-free asset or risky share.

Access to Treasury Bill

Extension: HH can access both treasury bill and risky share.

- Buy share s at price p, get return $(z' + p')$ next period
- Buy treasury bill a at price q, get return 1 next period

Budget constraint:

$$c + ps' + qa' \leq [z + p]s + a \tag{15}$$

Following the same approach before (blackboard!), we know the FOCs are

$$[a'] : \quad u'(c)q = \beta \mathbb{E}_{z'|z} \left[u'(c') \right] \Rightarrow q = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \right] \tag{16}$$

$$[s'] : \quad u'(c)p = \beta \mathbb{E}_{z'|z} \left[u'(c')z' + u'(c')p' \right] \tag{17}$$

No-arbitrage Condition

Household are indifferent to treasury bill or risky share in equilibrium.

Math representation can be observed by rearranging two FOCs:

$$[a']: \quad 1 = \mathbb{E}_{z'|z}\left[\beta \frac{u'(c')}{u'(c)} \times \frac{1}{q}\right] \tag{18}$$

$$[s']: \quad 1 = \mathbb{E}_{z'|z}\left[\beta \frac{u'(c')}{u'(c)} \times \frac{z'+p'}{p}\right] \tag{19}$$

Here, we derived expressions for the **rate of return**:

- **Gross return on share**: $e(z,z') \equiv \frac{z'+p'}{p}$
- **Gross return on risk-free bond**: $R(z) \equiv \frac{1}{q}$

Risk Premium: Derivation

From conditional covariance relation:

$$cov_z(A,B) = \mathbb{E}_{z'|z}\left[\left(A - \mathbb{E}_{z'|z}[A]\right)\left(B - \mathbb{E}_{z'|z}[B]\right)\right] \tag{20}$$

$$= \mathbb{E}_{z'|z}[AB] - \mathbb{E}_{z'|z}[A]\mathbb{E}_{z'|z}[B] \tag{21}$$

Equation (19) is $\mathbb{E}_{z'|z}[AB]$, where $A = \frac{\beta u'(c')}{u'(c)}$ and $B = e(z,z')$, and thus

$$1 = \mathbb{E}_{z'|z}\left[\beta \frac{u'(c')}{u'(c)} \times e(z,z')\right] \tag{22}$$

$$= cov_z\left[\beta \frac{u'(c')}{u'(c)}, e(z,z')\right] \tag{23}$$

$$+ \underbrace{\mathbb{E}_{z'|z}\left[\beta \frac{u'(c')}{u'(c)}\right]}_{\equiv \frac{1}{R(z)}\text{, return on bond}} \mathbb{E}_{z'|z}\left[e(z,z')\right] \tag{24}$$

Risk Premium: Derivation (Cont.)

Therefore

$$1 = cov_z \left[\frac{\beta u'(c')}{u'(c)}, e(z, z') \right] \tag{25}$$

$$+ \mathbb{E}_{z'|z}[e(z, z')] \times \frac{1}{R(z)} \tag{26}$$

$$\underbrace{\frac{\mathbb{E}_{z'|z}[e(z, z')] - R(z)}{R(z)}}_{\text{risk premium}} = -cov_z \left[\underbrace{\frac{\beta u'(c')}{u'(c)}}_{\text{SDF}}, e(z, z') \right] \tag{27}$$

Stochastic discounting factor (SDF) / pricing kernel: HH's discounted percentage change in marginal utility.

Risk Premium: Discussion

$$\underbrace{\frac{\mathbb{E}_{z'|z}[e(z,z')] - R(z)}{R(z)}}_{\text{risk premium}} = -cov_z\left[\underbrace{\frac{\beta u'(c')}{u'(c)}}_{\text{SDF}}, e(z,z')\right]$$

1. If return on share, $e(z,z')$, is uncorrelated with consumption change $c \to c'$, then risk premium is 0
   - HH not bearing risk $\Rightarrow$ no need to compensate HH
2. If return on share, $e(z,z')$, is high when c' is low, i.e., $e(z,z')$ and SDF are positively correlated, then risk premium is **negative**
   - HH is benefiting from buying this asset $\Rightarrow$ need to charge HH!
3. If return on share, $e(z,z')$, is low when c' is low, i.e., $e(z,z')$ and SDF are negatively correlated, then risk premium is **positive**
   - HH is bearing risk $\Rightarrow$ need to compensate HH

Appendix

References I

Lucas, Robert E. (1978) "Asset Prices in an Exchange Economy," *Econometrica*, 46 (6), 1429, 10.2307/1913837.