# Lecture 10
# Examples on Competitive Equilibrium and Social Planner's Problem

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# Overview

After constructing both consumers' and firms' problem, we start to bring them together in one-period model:

- Lecture 8: competitive equilibrium (CE)
  - each agent solve their problems individually
  - aggregate decision determines "prices" (wage, rent, etc.)
- Lecture 9: social planer's problem (SPP)
  - imaginary and benevolent social planner determines the allocation
  - should be the most efficient outcome
- Lecture 10: CE and SPP examples

# Outline

# Two Dimensional Chain Rule

Suppose we have a utility function $U(C, l)$, where $C$ is the consumption, and $l$ is the leisure, and both $C = C(w)$ and $l = l(w)$ are the function of equilibrium wage $w$, then

$$\begin{aligned} \frac{d}{dw}[U(C(w), l(w))] &= D_C U(C(w), l(w)) \times \frac{dC(w)}{dw} \\ &\quad + D_l U(C(w), l(w)) \times \frac{dl(w)}{dw} \end{aligned} \tag{1}$$

# "Taken as Given"

Here is a good rule of thumb:

When you solve the problem of an agent who chooses $y$ taking $x$ as given, the answer should take the form of $y(x)$.

**Example**: the consumer maximizes utility by choosing consumption, leisure, and labor supply, taking the wage and profits as given. $(G = 0)$

$$\max_{C,l,N^s} U(C,l) \quad \text{subject to} \quad C = wN^s + \pi \quad \text{and} \quad l + N^s = h \tag{2}$$

- solution takes the form: $C(w,\pi)$, $l(w,\pi)$, $N^s(w,\pi)$
- why not $h$, or utility parameters? Not **endogenous to the model!**
- can repeat this idea for the firm to get $N^d(w)$, $Y(w)$, $\pi(w)$

# "Endogenous to the Model"

What does equilibrium do? Figures out what level of "taken as given" but endogenous variables has to occur:

- consumer: $\pi = \pi(w)$ from firm's problem
- labor supply can be rewrite as: $N^s(w, \pi) = N^s(w, \pi(w)) = N^s(w)$
- labor market clearing: $N^d(w^*) = N^s(w^*)$, where $w^*$ is eqm wage

**Question**: any of the "taken as given variables" show up in the SPP?

- Ans: NO! Social planner is benevolent dictator!

# Outline

# Model Environment

- Consumer: $U(C,l) = \frac{C^{1-b}}{1-b} + \frac{l^{1-d}}{1-d}$, where $b = 2$ and $d = \frac{3}{2}$.
  - $b, d$ are parameters
  - $h = 1$ is time endowment to allocate between leisure and labor supply
  - owns the firm, subject to lump-sum tax $T \geq 0$
- Firm: $zF(K,N) = zK^{\alpha}N^{1-\alpha}$, where $K = 1$ and $\alpha = \frac{1}{2}$ (param)
- Government: $T = G$
- Labor market: both consumer and firm take wage rate $w$ as given

# Experiments

1. **Benchmark:** $z = 1$ and $G = 0$

2. **Experiment 1:** $z = 1.2$ and $G = 0$

3. **Experiment 2:** $z = 1$ and $G = 0.5$

# Outline

# Solve Benchmark in Social Planner's Problem

- PPF: $C + G = zN^{1-\alpha}$, where $\alpha = \frac{1}{2}$
- Time: $N = h - l$, where $h = 1$
- Social Planner's Problem:

$$\begin{aligned}
\max_{l} \quad & U(C(l), l) = \frac{C(l)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \\
\text{s.t.} \quad & C = Y - G \\
& Y = zN^{1-\alpha} \\
& N = 1 - l \\
\Rightarrow \quad \max_{l} \quad & \frac{(z(1-l)^{1-\alpha} - G)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d}
\end{aligned} \tag{3}$$

# Solve Benchmark in Social Planner's Problem (Cont.)

$$\max_{l} \quad \frac{(z(1-l)^{1-\alpha} - G)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \tag{4}$$

$$\text{FOC:} \quad \underbrace{(z(1-l)^{1-\alpha} - G)^{-b}}_{\frac{(\cdot)^{1-b}}{1-b}} \times \underbrace{(1-\alpha)z(1-l)^{-\alpha}}_{z(1-l)^{1-\alpha}} \times \underbrace{(-1)}_{-l} + l^{-d} = 0 \tag{5}$$

$$G = 0: \quad z^{-b}(1-l)^{-b(1-\alpha)} \times (1-\alpha)z(1-l)^{-\alpha} = l^{-d} \tag{6}$$

$$(1-\alpha)z^{1-b}(1-l)^{-\alpha-b+\alpha b} = l^{-d} \tag{7}$$

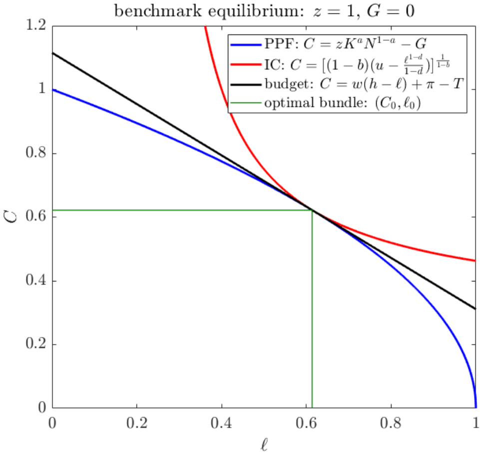
$$\alpha = 1/2; \quad b = 2; \quad d = 3/2 \tag{8}$$

$$\text{Apply:} \quad \frac{1}{2}z^{-1}(1-l)^{-\frac{3}{2}} = l^{-\frac{3}{2}} \Rightarrow \frac{1}{2z} = \left(\frac{1-l}{l}\right)^{\frac{3}{2}} \tag{9}$$

$$\Rightarrow \frac{1-l}{l} = \left(\frac{1}{2z}\right)^{\frac{2}{3}} \Rightarrow l(z,0) = \frac{1}{1+(2z)^{-\frac{2}{3}}} \tag{10}$$

$$z = 1 \quad \Rightarrow l \approx 0.61, N \approx 0.39, Y = C \approx 0.62, w = \frac{z}{2}N^{-\frac{1}{2}} \approx 0.8 \tag{11}$$

# Visualization: Benchmark in SPP

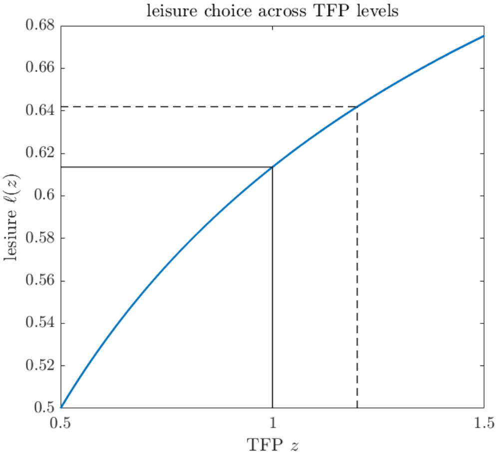



Indifference curve and PPF are tangent at optimal bundle

$$
\begin{aligned}
& \text{slope at tangency } (C_0, l_0) \\
= \quad & \text{slope of IC}(-MRS_{l,C}) \\
= \quad & \text{slope of budget line}(-w) \\
= \quad & \text{slope of PPF}(-MRT_{l,C}) \\
= \quad & \text{slope of production fcn}(-MPN)
\end{aligned}
$$

# Outline

# Solving with New TFP

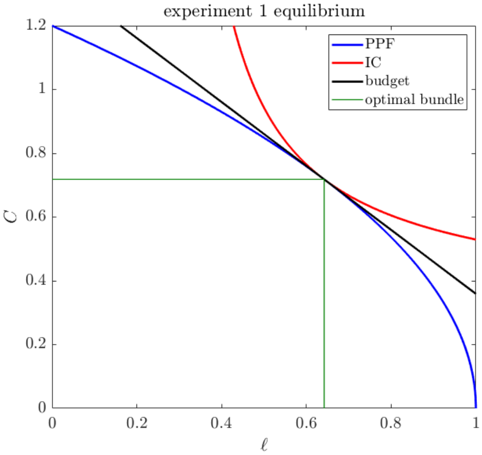



Recall that we solved for the equilibrium quantity of leisure as a function of TFP:

$$l(z) = \frac{1}{1 + (2z)^{-\frac{2}{3}}} \tag{12}$$

So now we' ve solved for all possible "experiment 1' s"! Just plug in $z = 1.2$ to get $l \approx 0.642$, and plug in to get all the rest as well.

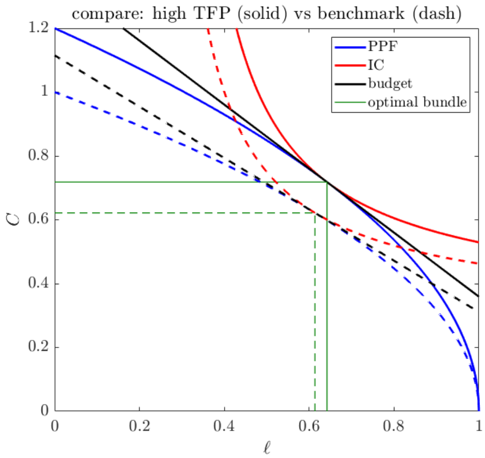
# Visualization: Experiment 1




Tangency preserved, just shifted

$$
\begin{aligned}
& \text{slope at tangency } (C_1, l_1) \\
= \quad & \text{slope of IC}(-MRS_{l,C}) \\
= \quad & \text{slope of budget line}(-w) \\
= \quad & \text{slope of PPF}(-MRT_{l,C}) \\
= \quad & \text{slope of production fcn}(-MPN)
\end{aligned}
$$

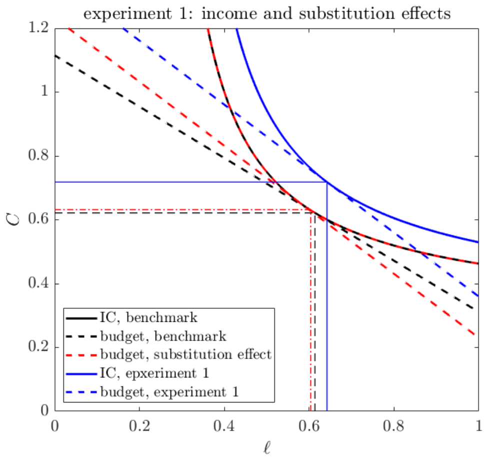
# Comparison: Experiment 1 and Benchmark




What' s different?

- higher productivity means PPF shifts outward
- outward shift of PPF makes higher utility level (IC) attainable
- tangency is steeper: wage increases
- both consumption and leisure increase!

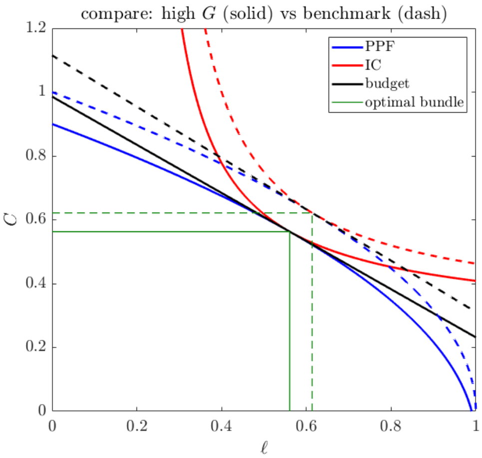
# Experiment 1: Income and Substitution Effect




Recall wage increase case from the consumer problem:

- **substitution effect:** move along IC but reflect new wage (i,e, new budget or new PPF)
  - $C$ increases, $l$ decreases
- **income effect:** move up to new budget line / PPF
  - $C$ and $l$ both increase
- here, income effect wins and leisure increases

# Comparison: Experiment 2 and Benchmark




Note: SPP harder to solve by hand with $G \neq 0$ ▸ details. But, can still analyze with graphs!

- higher government spending shifts PPF inward
- inward shift of PPF lowers utility level (IC) attainable
- budget shallower: wage falls
- consumption, leisure fall (recall normal goods assumption)
- can show output increases

# Outline

# Response to Data

| Effect of $\uparrow$ in | TFP | G |
|---|---|---|
| Output | Increase | Increase |
| Consumption | Increase | Decrease |
| Employment | Ambiguous | Increase |
| Wage | Increase | Decrease |

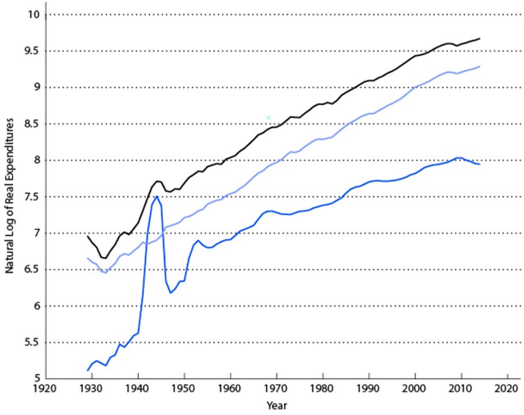
TFP is a overall better match!

Real Business Cycle theory

- recall key business cycle facts: employment, consumption, real wage are all procyclical
- recall key trend: output has grown steadily for last century
- question: which model is more consistent with these facts?

# Data: Government Spending from WWII

Figure: Figure 5.7 GDP, Consumption, and Government Expenditures




- large increase in $G$ to finance war effort
- modest increase in $Y$
- slight decline in $C$
- consistent with our model!

# Data: Solow Residual, $z = \frac{Y}{K^{\alpha}N^{1-\alpha}}$

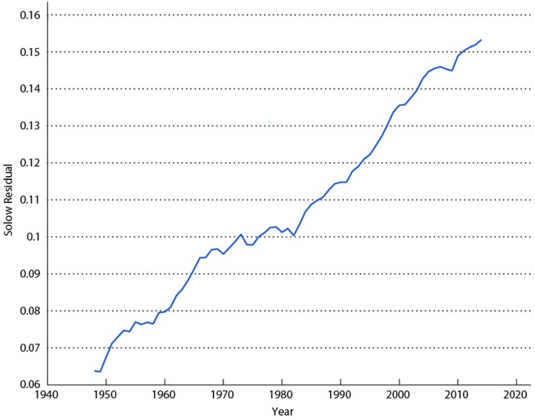



Figure: Figure 4.18 The Solow Residual for the United States

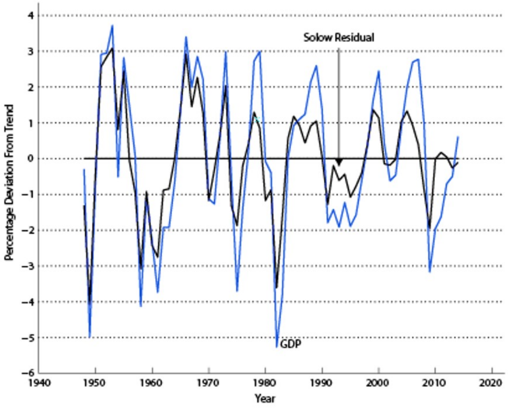



Figure: Figure 5.11 Deviations from Trend in GDP and the Solow Residual

# Outline

# Appendix

# How to solve $G \neq 0$

$$\max_{l} \quad \frac{(z(1-l)^{1-\alpha} - G)^{1-b}}{1-b} + \frac{l^{1-d}}{1-d} \tag{13}$$

$$\text{FOC:} \quad z(1-l)^{1-\alpha} - G)^{-b} \times (1-\alpha) z (1-l)^{-\alpha} = l^{-d} \tag{14}$$

$$\text{Divide :} \quad (z(1-l)^{1-\alpha} - G)^{-b} = \frac{l^{-d}}{(1-\alpha) z (1-l)^{-\alpha}} \tag{15}$$

$$\text{power of } -\frac{1}{b} : \quad z(1-l)^{1-\alpha} - G = \left[ \frac{l^{-d}}{(1-\alpha) z (1-l)^{-\alpha}} \right]^{-\frac{1}{b}} \tag{16}$$

$$\text{Solve } G : \quad G = F(l) = z(1-l)^{1-\alpha} - \left[ \frac{l^{-d}}{(1-\alpha) z (1-l)^{-\alpha}} \right]^{-\frac{1}{b}} \tag{17}$$

$$\iff l = F^{-1}(G) \tag{18}$$