

Asset Pricing in Endowment Economy

Hui-Jun Chen

National Tsing Hua University

April 14, 2026

How to value an asset?

- Lucas (1978) answer this question by the “fruit tree” story:
 - each representative household initially owns **one** fruit tree,
 - the fruits (dividend) from tree is **uncertain** each period, and
 - household **cannot store** the fruits.
- To achieve intertemporal substitution, HH can exchange the **property right** of the tree \Rightarrow **share** of the tree.
 - Each tree is identical \Rightarrow the randomness of fruit is identical.
 - If fruits (dividend) vary over time, how do share price varies?
 - What is the **fundamental value** of an asset?
 - How to compare fundamental value between **different assets**?

Value of Share

Model Setting

Let

- z be the fruit/dividend per tree
- s be the share of a tree,

Household's problem:

$$V(s, z) = \max_{s', c} u(c) + \beta \mathbb{E}_{z'|z} [V(s', z')] \quad (1)$$

$$\text{subject to } c + ps' \leq (z + p)s \quad (2)$$

Equilibrium outcome is trivial: since the only resource is z , so goods market clearing condition is $c = z$, which leads to $s = 1$ in equilibrium.

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😊 The End 😊...?

What is share prices?

- **Fundamental value of an asset** is the **expected discounted NPV** of all its future payoff.
- In the case of firm share the payoff is the dividend.
- and we are going to derive the share price by solving the **optimal choice of share**.

Recall HH's problem at time t can be written as

$$V(s, z) = \max_{s', c} u(c) + \beta \mathbb{E}_{z'|z}[u(c')] + \beta^2 \mathbb{E}_{z''|z}[V(s'', z'')] \quad (3)$$

$$\text{subject to } c + ps' \leq (z + p)s \quad (4)$$

$$c' + p's'' \leq (z' + p')s' \quad (5)$$

Optimal Choice of Share

Substitute,

$$\begin{aligned} V(s, z) = \max_{s'} & u((z + p)s - ps') \\ & + \beta \mathbb{E}_{z'|z} [u((z' + p')s' - p's'')] \\ & + \beta^2 \mathbb{E}[V(s'', z'')] \end{aligned}$$

FOC,

$$[s'] : \quad u'(c)p = \beta \mathbb{E}_{z'|z} [u'(c')(z' + p')] = \beta \mathbb{E}_{z'|z} [u'(c')z' + u'(c')p'] \quad (6)$$

Notice that the current price p depends on future price p' !

Fundamental Value of Share: Euler Equation Update

Nothing stops us to figure out what p' would look like, and it should be update (6) one period forward:

$$u'(c')p' = \beta \mathbb{E}_{z''|z'} [u'(c'')(z'' + p'')]]$$

So (6) will be

$$u'(c)p = \mathbb{E}_{z'|z} [\beta u'(c')z' + \beta^2 \mathbb{E}_{z''|z'} [u'(c'')(z'' + p'')]] \quad (7)$$

$$= \mathbb{E}_{z'',z'|z} [\beta u'(c')z' + \beta^2 u'(c'')z'' + \beta^2 u'(c'')p''] \quad (8)$$

Repeat substitution and get

$$u'(c)p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j})z_{t+j} \right] + \lim_{j \rightarrow \infty} \beta^j u'(c_{t+j})p_{t+j} \quad (9)$$

Fundamental Value of Share

By transversality condition,

$$\lim_{j \rightarrow \infty} \beta^j u'(c_{t+j}) p_{t+j} = 0 \quad (10)$$

and thus,

$$u'(c) p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j u'(c_{t+j}) z_{t+j} \right] \quad (11)$$

with goods market clearing condition $c_t = z_t$,

$$p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j} \right] \quad (12)$$

Fundamental Value of Share: Analysis

$$p_t = \mathbb{E}_t \left[\sum_{j=1}^{\infty} \beta^j \frac{u'(z_{t+j})}{u'(z_t)} z_{t+j} \right] \quad (13)$$

Take one of the period, say $j = 3$, the **evaluation** of dividend z_{t+3} is

$$\underbrace{\beta^3 \frac{u'(z_{t+3})}{u'(z_t)}}_{\text{subjective discounted evaluation of dividend}} \underbrace{z_{t+3}}_{\text{objective flow dividend}} \quad (14)$$

Thus, we show that the share price p represents the **expectation** of **subjective discounted evaluation** of all future dividends.

Risk Premium

Comparison between Evaluations

How to compare the value of different assets?

- **Risk premium:** measure of excess return that is required by an individual to compensate being subjected to an **increased level of risk**.
 - i.e., the **percentage compensation** required for household to take risk.
 - math:
$$\frac{\text{return on share} - \text{return on risk-free}}{\text{return on risk-free}}$$
- Share is one kind of risky asset: the fruits from the tree is random.
- If household is able to access to **risk-free asset**, in equilibrium household should be **indifferent** between buying risk-free asset or risky share.

Access to Treasury Bill

Extension: HH can access both treasury bill and risky share.

- Buy share s at price p , get return $(z' + p')$ next period
- Buy treasury bill a at price q , get return 1 next period

Budget constraint:

$$c + ps' + qa' \leq [z + p]s + a \quad (15)$$

Following the same approach before (blackboard!), we know the FOCs are

$$[a'] : u'(c)q = \beta \mathbb{E}_{z'|z} [u'(c')] \Rightarrow q = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \right] \quad (16)$$

$$[s'] : u'(c)p = \beta \mathbb{E}_{z'|z} [u'(c')z' + u'(c')p'] \quad (17)$$

No-arbitrage Condition

Household are indifferent to treasury bill or risky share in equilibrium.
Math representation can be observed by rearranging two FOCs:

$$[a'] : 1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times \frac{1}{q} \right] \quad (18)$$

$$[s'] : 1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times \frac{z' + p'}{p} \right] \quad (19)$$

Here, we derived expressions for the **rate of return**:

- **Gross return on share:** $e(z, z') \equiv \frac{z' + p'}{p}$
- **Gross return on risk-free bond:** $R(z) \equiv \frac{1}{q}$

Risk Premium: Derivation

From [conditional covariance relation](#):

$$\text{cov}_z(A, B) = \mathbb{E}_{z'|z} \left[(A - \mathbb{E}_{z'|z}[A]) (B - \mathbb{E}_{z'|z}[B]) \right] \quad (20)$$

$$= \mathbb{E}_{z'|z} [AB] - \mathbb{E}_{z'|z}[A]\mathbb{E}_{z'|z}[B] \quad (21)$$

Equation (19) is $\mathbb{E}_{z'|z} [AB]$, where $A = \frac{\beta u'(c')}{u'(c)}$ and $B = e(z, z')$, and thus

$$1 = \mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \times e(z, z') \right] \quad (22)$$

$$= \text{cov}_z \left[\beta \frac{u'(c')}{u'(c)}, e(z, z') \right] \quad (23)$$

$$+ \underbrace{\mathbb{E}_{z'|z} \left[\beta \frac{u'(c')}{u'(c)} \right]}_{\equiv \frac{1}{R(z)}, \text{ return on bond}} \mathbb{E}_{z'|z} [e(z, z')] \quad (24)$$

Risk Premium: Derivation (Cont.)

Therefore

$$1 = cov_z \left[\frac{\beta u'(c')}{u'(c)}, e(z, z') \right] \quad (25)$$

$$+ \mathbb{E}_{z'|z}[e(z, z')] \times \frac{1}{R(z)} \quad (26)$$

$$\underbrace{\frac{\mathbb{E}_{z'|z}[e(z, z')] - R(z)}{R(z)}}_{\text{risk premium}} = -cov_z \left[\underbrace{\frac{\beta u'(c')}{u'(c)}}_{\text{SDF}}, e(z, z') \right] \quad (27)$$

Stochastic discounting factor (SDF) / pricing kernel: HH's discounted percentage change in marginal utility.

$$\underbrace{\frac{\mathbb{E}_{z'|z}[e(z, z')] - R(z)}{R(z)}}_{\text{risk premium}} = -\text{cov}_z \left[\underbrace{\frac{\beta u'(c')}{u'(c)}}_{\text{SDF}}, e(z, z') \right]$$

1. If return on share, $e(z, z')$, is **uncorrelated** with consumption change $c \rightarrow c'$, then risk premium is **0**
 - HH not bearing risk \Rightarrow no need to compensate HH
2. If return on share, $e(z, z')$, is high when c' is low, i.e., $e(z, z')$ and SDF are **positively correlated**, then risk premium is **negative**
 - HH is benefiting from buying this asset \Rightarrow need to charge HH!
3. If return on share, $e(z, z')$, is low when c' is low, i.e., $e(z, z')$ and SDF are **negatively correlated**, then risk premium is **positive**
 - HH is bearing risk \Rightarrow need to compensate HH