# Graduate Macro Sequence: Three ways to represent a model

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# Three ways to represent a model

- Same economics, three "lenses":
  1. Date-0 (Arrow–Debreu / planning): choose an entire plan at time 0
  2. Sequential (markets each period): choose each period subject to a per-period budget
  3. Recursive (dynamic programming): choose a *rule* using a state variable
- Key message for beginners:

$$\textbf{Plan} \quad (\{c_t, k_{t+1}\}_{t\geq 0}) \quad \Longleftrightarrow \quad \textbf{Rule} \quad (c = g(x),\ k' = h(x))$$

- We use the same neoclassical growth environment as an example

# Roadmap

1. Warm-up: Plan vs Rule in a 2-period problem
2. Infinite horizon: why "plan form" becomes an infinite-dimensional object
3. The three representations with Neoclassical Growth Model as an example
4. Why recursion is computationally powerful (fixed point / iteration)

# Warm-up: plan vs rule

# Warm-up: a 2-period consumption–saving problem

Suppose you live for two periods $t = 0, 1$.

- Resources:

$$c_0 + a_1 = y_0 + (1+r)a_0, \qquad c_1 = y_1 + (1+r)a_1.$$

- Preferences:

$$u(c_0) + \beta u(c_1), \quad \beta \in (0,1).$$

**Two ways to think about it:**

1. Plan: choose $(c_0, a_1, c_1)$ today.

2. Rule: choose $a_1$ today, and tomorrow consume whatever is feasible.

# The only new idea: continuation value

At $t = 0$, write the objective as

$$u(c_0) \ + \ \beta \qquad \underbrace{u(c_1)}_{\text{everything after today}} \qquad .$$

In longer horizons, "everything after today" is a long tail:

$$u(c_0) + \beta u(c_1) + \beta^2 u(c_2) + \cdots$$

- DP names this tail: continuation value.
- Continuation value depends on what you carry into tomorrow:

$$\text{tomorrow's situation} \quad \approx \quad \text{state}.$$

**Transition:** For infinite horizon, we cannot treat the tail as "a finite list." We compress it into a function $V(\cdot)$.

# Why infinite horizon motivates recursion

# Why infinite horizon is hard in plan form

Date-0 / sequential formulations ask you to pick an entire sequence:

$$\{c_t, k_{t+1}\}_{t=0}^{\infty}.$$

- That is an infinite-dimensional object.
- You can derive elegant conditions (Euler equation, transversality conditions),
- But you still need a way to compute the policy rules.

**DP's goal:** replace an infinite sequence with two functions:

$$c_t = g(x_t), \qquad x_{t+1} = f(x_t, g(x_t), \varepsilon_{t+1}).$$

# What is a state?

A state variable $x_t$ is a summary of "where you are" today that is sufficient for:

1. choosing optimally today,
2. predicting the distribution of tomorrow.

**In the neoclassical growth model (no labor):**

$$x_t = k_t \quad \text{(current capital)}.$$

Given $k_t$, today's choice $k_{t+1}$ pins down today's consumption:

$$c_t = f(k_t) + (1-\delta)k_t - k_{t+1}.$$

- Past history matters only through $k_t$.
- So the optimal decision can be written as a rule:

$$k_{t+1} = h(k_t), \qquad c_t = g(k_t).$$

# Three Representation

# Neoclassical Growth Model: Set up

- Micro-foundation: rep. consumer makes consumption-saving decision.
- No externalities, and thus can solve in Social planner's problem.
- Assume rep. consumer lives for $\infty$ period with **additive** separability:

$$U(C_0, C_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(C_t), \tag{1}$$

  where function $u(\cdot)$ is the same for every period, and $\beta$ is subjective discount factor.
- Assumes no labor (for the sake of sanity)
- Two goods are trading:
    - firm $\rightarrow$ consumer: consumption goods ($c_t$) with price $p_t$
    - consumer $\rightarrow$ firm: capital accumulation ($k_t$) with price $r_t$

# Date 0 Representation

A Date 0 C.E. is **prices** $\{p_t, r_t\}_{t=0}^{\infty}$ and **quantities** $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ such that

1. $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves household's problem,

$$\max_{\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2}$$

$$\text{subject to} \quad c_t \geq 0, \forall t = 0, 1, \ldots \tag{3}$$

$$\sum_{t=0}^{\infty} p_t(c_t + k_{t+1}) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + (1-\delta)k_t), \forall t \tag{4}$$

2. $\{k_{t+1}^*\}_{t=0}^{\infty}$ solves firm's problem at each $t = 0, 1, \ldots$

$$\max_{k_t} p_t f(k_t) - p_t r_t k_t \tag{5}$$

3. Goods market clear: $c_t^* + k_{t+1}^* = f(k_t^*) + (1-\delta)k_t^*$

# Discussion on Date 0 Representation

- $p_t$ is the relative price of $c_t$ **in units of** $c_0 \Rightarrow p_0 = 1$.
- $p_t r_t$ is the relative price of capital **in units of** $c_0$
- Firm's problem is static, implies $r_t = D_k f(k_t)$
- Use **LaGrange multiplier** $\lambda$, we derive the FOC for $c_t$ and $k_{t+1}$ are

$$\begin{aligned}[c_t]: &\quad \beta^t u'(c_t) = \lambda p_t \\ [k_{t+1}]: &\quad p_t = p_{t+1}(r_{t+1} + 1 - \delta)\end{aligned}$$

- If we divide both $p_t$ and $p_{t+1}$, we get **Euler equation**:

$$\frac{p_t}{p_{t+1}} = \frac{u'(c_t)}{\beta u'(c_{t+1})} = (r_{t+1} + 1 - \delta) \Rightarrow u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta)$$

# Sequential Representation

A sequential C.E. is **prices** $\{r_t\}_{t=0}^{\infty}$ and **quantities** $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ such that

1. $\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}$ solves household's problem,

$$\max_{\{c_t^*, k_{t+1}^*\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{6}$$

$$\text{subject to} \quad c_t \geq 0, \forall t = 0, 1, \ldots \tag{7}$$

$$c_t + k_{t+1} \leq r_t k_t + (1 - \delta) k_t, \forall t = 0, 1, \ldots \tag{8}$$

$$\lim_{t \to \infty} \left( \prod_{s=1}^{t} (r_t + 1 - \delta) \right)^{-1} k_{t+1} = 0 \tag{9}$$

2. $\{k_{t+1}^*\}_{t=0}^{\infty}$ solves firm's problem at each $t = 0, 1, \ldots$

$$\max_{k_t} f(k_t) - r_t k_t \tag{10}$$

3. Goods market clear: $c_t^* + k_{t+1}^* = f(k_t^*) + (1 - \delta) k_t^*$

# Discussion on Sequential Representation

- Here we have budget constraint at every possible $t$, rather than one.
- Need **LaGrange multiplier** $\lambda_t$ for each budget constraint!
- FOC for $c_t$ and $k_{t+1}$ are

$$\begin{aligned} [c_t]: &\quad \beta^t u'(c_t) = \beta^t \lambda_t \Rightarrow u'(c_t) = \beta \lambda_t \\ [k_{t+1}]: &\quad \beta^t \lambda_t = \beta^{t+1} \lambda_{t+1}(r_{t+1} + 1 - \delta) \Rightarrow \lambda_t = \beta \lambda_{t+1}(r_{t+1} + 1 - \delta) \end{aligned}$$

- and still, we can the same **Euler equation**:

$$u'(c_t) = \beta u'(c_{t+1})(r_{t+1} + 1 - \delta)$$

- Equation (9) is the transversality condition: avoid Ponzi scheme

# Motivating Recursive Representation

- In the sequential representation, at each date $t$, household is solving **exactly the same** utility optimization problem, so we can write it as:

$$\max_{c_t, k_{t+1}} u(c_t) + \overbrace{\sum_{s=t+1}^{\infty} \beta^s u(c_s)}^{\text{not related to } c_t} \tag{11}$$

$$\text{subject to} \quad c_t + k_{t+1} \leq r_t k_t + (1-\delta) k_t \tag{12}$$

$$c_{t+1} + k_{t+2} \leq r_{t+1} k_{t+1} + (1-\delta) k_{t+1} \tag{13}$$

- Observing this, instead of finding the **level** of the prices and quantities, we find the **function** of prices and quantities that express the same problem that household is solving **at each** $t$.

- Note that HH cannot change prices, and thus prices depends on the **aggregate** state variable, i.e., aggregate capital $\bar{K}$. In equilibrium $\bar{K} = k$.

# Recursive Representation

A recursive C.E. is a set of functions for **prices** $\{r(\bar{K})\}$ and **quantities** $\{G(\bar{K}), g(k, \bar{K})\}$ and value $V(k, \bar{K})$ such that

1. $V(k, \bar{K})$ solves household's problem,

$$V(k, \bar{K}) = \max_{c, k' \geq 0} (u(c) + \beta V(k', \bar{K}')) \tag{14}$$

$$\text{subject to} \quad c + k' = (r(\bar{K}) + 1 - \delta)k \tag{15}$$

$$\bar{K}' = G(\bar{K}) \tag{16}$$

2. Prices are competitively determined, i.e., firm's problem implies

$$r(\bar{K}) = f'(\bar{k}),$$

3. Individual decisions are consistent with aggregates when $k = \bar{K}$, i.e.,

$$G(\bar{K}) = g(\bar{K}, \bar{K})$$

# Why recursion is solvable/computable

# Why the recursive form is computable

The recursive form turns the problem into a fixed point:

$$V = T(V),$$

where $T$ is the Bellman operator:

$$(TV)(k) = \max_{k' \in \Gamma(k)} \left\{ u(f(k) + (1-\delta)k - k') + \beta V(k') \right\}.$$

- Start with a guess $V_0$
- Update: $V_{n+1} = T(V_n)$
- Repeat until $V_{n+1} \approx V_n$

**Interpretation:** "Solve the same two-period problem again and again."

# One model, three representations

- We keep the same primitives and same feasibility (neoclassical growth).
- What changes is the equilibrium object we solve for:
    - Date-0: sequences of prices $\{p_t, r_t\}_{t\geq 0}$ and allocations $\{c_t, k_{t+1}\}_{t\geq 0}$
    - Sequential: spot prices $\{r_t\}_{t\geq 0}$, allocations, and a TVC
    - Recursive: functions $r(\bar{K})$, policies $g(k, \bar{K})$, aggregation $G(\bar{K})$, value $V(k, \bar{K})$

| Date-0 CE | Sequential CE | Recursive CE |
|---|---|---|
| One PV constraint | Per-period constraints<br>Transversality condition | Bellman + consistency |
| Prices: $\{p_t, r_t\}$<br>Allocations: sequences | Prices: $\{r_t\}$<br>Allocations: sequences | Prices: $r(\bar{K})$<br>Allocations: policy rules |

# Takeaway

1. A model can be written as plan, sequential, or recursive.
2. They describe the same economics but emphasize different objects.
3. DP is the step that turns “choose an infinite sequence” into “compute a rule.”

$$\text{Plan} \iff \text{Rule}$$