Graduate Macro Sequence: One-Side Labor Search Model

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One-Sided search [McCall (1970)]

decision maker = unemployed worker

- infinitely lived and risk neutral
- no borrowing/lending
- no aggregate uncertainty
- lifetime utility $\sum_{t=0}^{\infty} \beta^t u(c_t)$, where $u(c_t) = c_t$
- One i.i.d. wage offer w drawn each period while unemployed
- $w \sim F(w)$ defined on $[0, B]$, with $B \in \mathbb{R}_+$
- Accept $\rightarrow$ get w each period forever, starting now
- Reject $\rightarrow$ get c (UI insurance, etc) now; get new draw next period

One-Sided search [McCall (1970)]

unemployed worker's problem

- define $v(w)$ as expected lifetime utility value of receiving offer w

- simple recursive formulation of problem:

$$v(w) = \max\{V^{accept}(w),\ V^{reject}\}$$

$$V^{accept}(w) = \sum_{t=0}^{\infty} \beta^t w = \frac{w}{1-\beta} \qquad V^{reject} = c + \beta \int_0^B v(w')F(dw')$$

- consolidated value function for an unemployed worker:

$$v(w) = \max\left\{\frac{w}{1-\beta},\ c + \beta \int_0^B v(w')F(dw')\right\}$$

One-Sided search [McCall (1970)]

Value for a worker with offer w in hand

$$v(w) = \max\left\{\frac{w}{1-\beta},\ c + \beta \int_0^B v(w')F(dw')\right\}$$

- reservation wage:

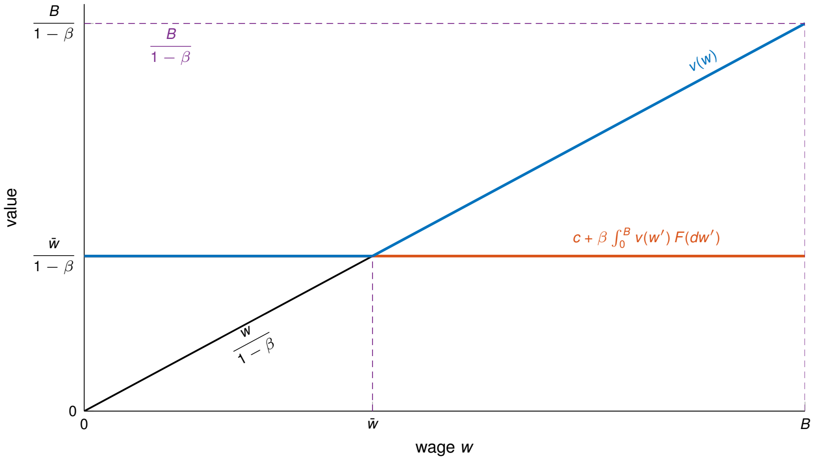
$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(w')F(dw') \tag{1}$$

- restated functional equation:

$$v(w) = \begin{cases} \dfrac{\bar{w}}{1-\beta}, & w \le \bar{w} \\ \dfrac{w}{1-\beta}, & w \ge \bar{w} \end{cases} \tag{2}$$

- *result 1*: reservation wage rises in unemployment compensation, c.

First look at the reservation wage



- *result 1:* reservation wage rises in unemployment compensation, c.
- $v(w)$ also rises in c [weakly].

First Characterization

Characterizing the reservation wage (ver. 1)

$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^B v(w')F(dw') \qquad \text{with} \qquad v(w) = \begin{cases} \frac{\bar{w}}{1-\beta}, & w \leq \bar{w} \\ \frac{w}{1-\beta}, & w \geq \bar{w} \end{cases}$$

$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^{\bar{w}} \frac{\bar{w}}{1-\beta} F(dw') + \beta \int_{\bar{w}}^B \frac{w'}{1-\beta} F(dw')$$

$$\frac{\bar{w}}{1-\beta} = c + \frac{\beta}{1-\beta}\bar{w} + \frac{\beta}{1-\beta} \int_{\bar{w}}^B (w' - \bar{w}) F(dw')$$

$$\bar{w} - c = \frac{\beta}{1-\beta} \int_{\bar{w}}^B (w' - \bar{w}) F(dw') \tag{3}$$

Graphing the reservation wage (ver. 1)

$$\bar{w} - c = \frac{\beta}{1-\beta}\int_{\bar{w}}^{B}(w' - \bar{w})F(dw') \tag{3}$$

reservation wage solves $w - c = h(w)$, where

$$h(w) \equiv \frac{\beta}{1-\beta}\int_{w}^{B}(w' - w)F(dw') \tag{4}$$

$$h(0) \equiv \frac{\beta}{1-\beta}\int_{0}^{B}(w')F(dw') = \frac{\beta}{1-\beta}Ew \qquad h(B) \equiv \frac{\beta}{1-\beta}\int_{B}^{B}(w' - B)F(dw') = 0$$

Graphing the reservation wage (ver. 1)

res. wage solves $w - c = h(w)$, with $h(w) \equiv \frac{\beta}{1-\beta} \int_w^B (w' - w) F(dw')$

recall Leibniz:

$$\frac{\partial}{\partial z}\left[\int_{a(z)}^{b(z)} f(x,z)\,dx\right] = \int_{a(z)}^{b(z)} \left[\frac{\partial f(x,z)}{\partial z}\right] dx + b'(z) f(b(z),z) - a'(z) f(a(z),z)$$

this application: $[w' - w]F'(w')dw'$ is $f(x,z)dx$ and w is z and w' is x

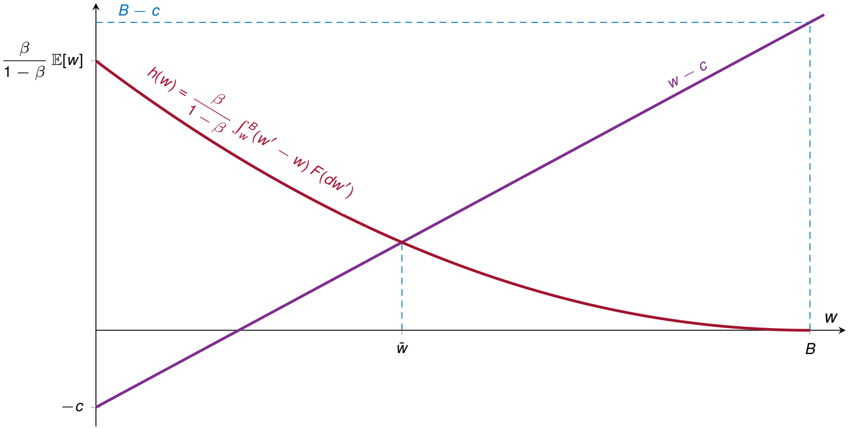
$$h'(w) = \frac{\beta}{1-\beta}\left[\int_w^B -F'(w')\,dw' + [0 \cdot (B-w)] - [1 \cdot (w-w)]\right]$$

$$h'(w) = -\frac{\beta}{1-\beta}[F(B) - F(w)] = -\frac{\beta}{1-\beta}[1 - F(w)] < 0 \quad \text{for } w < B$$

$$h''(w) = \frac{\beta}{1-\beta} F'(w) > 0 \quad \text{for } w < B$$

- RHS of wage equation, $w - c = h(w)$, is downward sloping and convex

Second look at the reservation wage



- higher β pivots $h(w)$ upward, with no effect on LHS.
- *result* 2: reservation wage rises with patience, β.

Mean-Preserving Spread

Definition Part A: Let $F(x)$ and $H(x)$ be two distributions on $x \in [0, B]$ with the same mean...

$$Ex = \int_0^B xF(dx) = \int_0^B xH(dx)$$

recall $\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$ and set $u = x, dv = F(dx)$:

$$\int_0^B xF(dx) = xF(x)\big|_0^B - \int_0^B F(x)\,dx = B - \int_0^B F(x)\,dx$$

$$\text{Common Mean:} \quad Ex = B - \int_0^B F(x)\,dx = B - \int_0^B H(x)\,dx$$

$$\text{Common Mean:} \quad \int_0^B [F(x) - H(x)]\,dx = 0 \tag{5a}$$

Mean-Preserving Spread

Definition Part B: And let F and H have the single crossing property...

$$\text{Single Crossing: } \exists \hat{x} \in (0, B) \text{ such that } F(x) - H(x) \begin{cases} \geq 0 & \text{at } x \leq \hat{x} \\ \leq 0 & \text{at } x \geq \hat{x} \end{cases} \tag{5b}$$

Definition Part C: Then F is a Mean-Preserving Spread of H.

- combining (5b) and (5a):

$$0 = \int_0^B [F(x) - H(x)]dx = \int_0^{\hat{x}} [F(x) - H(x)]dx + \int_{\hat{x}}^B [F(x) - H(x)]dx$$

Implication: If F is a Mean-Preserving Spread of H, then:

$$\int_0^y [F(x) - H(x)]dx \geq 0 \quad \forall y \in [0, B] \tag{5c}$$

Second Characterization

Characterizing the reservation wage (ver. 2)

- an alternative expression, beginning with equation (3)

$$\bar{w} - c = \frac{\beta}{1-\beta}\int_{\bar{w}}^{B}(w' - \bar{w})F(dw') + \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})F(dw') - \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})F(dw')$$

$$\bar{w} - c = \frac{\beta}{1-\beta}\int_{0}^{B}(w' - \bar{w})F(dw') - \frac{\beta}{1-\beta}\int_{0}^{\bar{w}}(w' - \bar{w})F(dw')$$

$$\bar{w} - (1-\beta)c = \beta\int_{0}^{B} w'F(dw') - \beta\int_{0}^{\bar{w}}(w' - \bar{w})F(dw')$$

$$\bar{w} - c = \beta(Ew - c) - \beta\int_{0}^{\bar{w}}(w' - \bar{w})F(dw') \tag{**}$$

Characterizing the reservation wage (ver. 2)

- so far:

$$\bar{w} - c = \beta(Ew - c) - \beta \int_0^{\bar{w}} (w' - \bar{w}) F(dw') \qquad (**)$$

- $\int_a^b u\,dv = uv\big|_a^b - \int_a^b v\,du$. Set $u = [w' - \bar{w}]$, $dv = F'(w')dw'$.

$$\int_0^{\bar{w}} (w' - \bar{w}) F(dw') = (w' - \bar{w}) F(w')\big|_0^{\bar{w}} - \int_0^{\bar{w}} F(w')dw' = -\int_0^{\bar{w}} F(w')dw'$$

- replacing the integral in (**), we can identify $\bar{w}$ via:

$$\bar{w} - c = \beta(Ew - c) + \beta \int_0^{\bar{w}} F(w')dw' \qquad (6)$$

Graphing the reservation wage (ver. 2)

- From (6), we know $\bar{w}$ is the w satisfying:

$$w - c = \beta(Ew - c) + \beta \int_0^w F(w')dw'$$

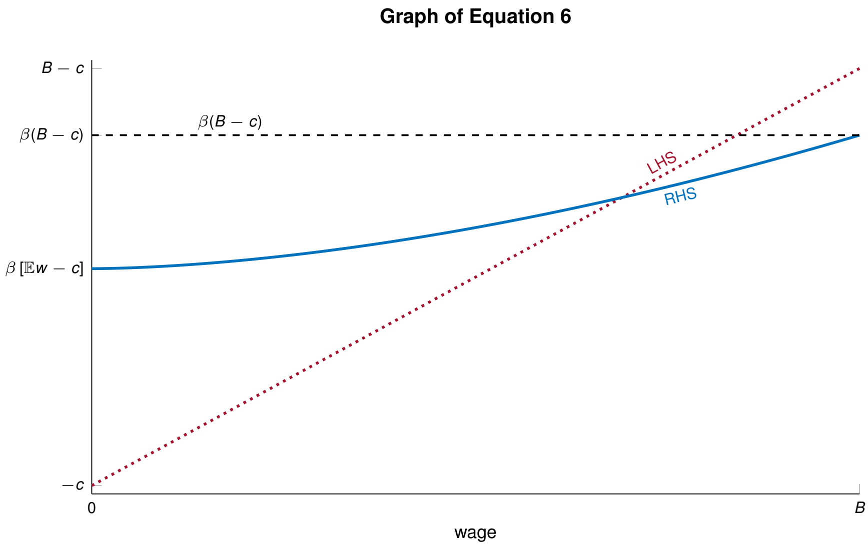
- LHS is linear, starts at $-c$, with slope 1.
- Term 1 of RHS is a constant $\geq -c$.
- Term 2 of RHS is $g(w) \equiv \beta \int_0^w F(w')dw'$.
- $g(0) = 0. \quad g'(w) = \beta F(w) \geq 0. \quad g''(w) = \beta F'(w) \geq 0.$

Leibniz Rule:

$$\frac{\partial}{\partial z}\left[\int_{a(z)}^{b(z)} f(x,z)\,dx\right] = \int_{a(z)}^{b(z)} \left[\frac{\partial f(x,z)}{\partial z}\right] dx + b'(z)f(b(z),z) - a'(z)f(a(z),z)$$

$$g'(w) = \beta\frac{\partial}{\partial w}\left[\int_0^w F(w')dw'\right] = \beta\left[\int_0^w \frac{\partial F(w')}{\partial w}\,dw' + 1 \cdot F(w) - 0 \cdot F(0)\right] = \beta F(w)$$

Graph of Equation (6)



Graphing the reservation wage (ver. 2)

$\bar{w}$ is the w satisfying: $\quad w - c = \beta(Ew - c) + \beta \int_0^w F(w')dw'.$

What if F is replaced by G, a Mean Preserving Spread of F?

- Linear LHS unchanged. Constant Term 1 of RHS unchanged.
- Increasing, convex Term 2 of RHS,

$$g_F(w) \equiv \beta \int_0^w F(w')dw', \quad \text{becomes} \quad g_G(w) \equiv \beta \int_0^w G(w')dw'.$$

- Prior results on Mean Preserving Spreads tell us

$$\int_0^y [G(w') - F(w')]dw' > 0 \quad \forall y \in [0, B). \tag{5c}$$

- *result* 3: reservation wage rises with a MPS on wage offers.

Hazard and Unemployment Duration

A. Hazard: Start-of-period employment probability (pre-w-draw)

$$H_t = \Pr(accept) = 1 - \Pr(reject) = 1 - F(\bar{w})$$

B. Unemployment Spell of Duration $D = d$ occurs when $d-1$ offers are rejected and subsequent offer is accepted.[1]

- $\Pr(D = 1) = H_t$
- For $d > 1$: $\Pr(D = d) = (1 - H_t)(1 - H_{t+1}) \cdots (1 - H_{t+d-2}) \cdot H_{t+d-1}$
- All cases, $d \geq 1$:

$$\Pr(D = d) = (1 - H)^{d-1} \cdot H = [F(\bar{w})]^{d-1}[1 - F(\bar{w})]$$

[1] Given this definition (adopting the convention in the search literature), we identify immediate job acceptance as an unemployment spell of duration 1. Thus, unemployment durations run $d = 1, 2, \ldots$

Hazard and Unemployment Duration

A. Hazard: Start-of-period employment probability (pre-w-draw)

$$H_t = \Pr(accept) = 1 - \Pr(reject) = 1 - F(\bar{w})$$

B. Unemployment Spell of Duration $D = d$ occurs when $d - 1$ offers are rejected and subsequent offer is accepted.

$$\Pr(D = d) = [F(\bar{w})]^{d-1}[1 - F(\bar{w})], \quad \text{for all } d \geq 1$$

C. Expected Unemployment Duration:

$$ED = \sum_{d=1}^{\infty} d \cdot (1 - H)^{d-1} \cdot H$$

Expected Unemployment Duration

Expected Duration:

$$ED = \sum_{d=1}^{\infty} d \cdot (1-H)^{d-1} \cdot H$$

A Handy Trick:

$$\sum_{d=0}^{\infty} \Pr(D=d) = \sum_{d=0}^{\infty} (1-H)^{d} \cdot H \equiv 1.$$

Differentiate w.r.t. H:

$$\sum_{d=0}^{\infty} d(1-H)^{d-1} \cdot (-H) + \sum_{d=0}^{\infty} (1-H)^{d} = 0$$

$$\Rightarrow \sum_{d=0}^{\infty} d(1-H)^{d-1} \cdot H = \frac{1}{1-(1-H)} \quad [d=0 \text{ contributes 0 on LHS}]$$

$$\Rightarrow \sum_{d=1}^{\infty} d(1-H)^{d-1} \cdot H = \frac{1}{H}$$

Expected Unemployment Duration: $ED = \frac{1}{H} = [1-F(\bar{w})]^{-1}.$

Influence of c and $F(w)$ on Duration and Welfare

- Worker's optimal policy is a stationary reservation wage strategy, with reservation wage implicitly defined by (6).

$$\bar{w} - c = \beta(Ew - c) + \beta \int_0^{\bar{w}} F(w')dw' \tag{6}$$

- Raised c or MPS of $F \Rightarrow \bar{w} \uparrow \Rightarrow F(\bar{w}) \uparrow \Rightarrow H \downarrow \Rightarrow ED \uparrow$

- Longer unemployment duration does NOT imply the worker is hurt by $c \uparrow$ or MPS in wage offers. Quite the converse.

$$Ev = \frac{\bar{w}}{1-\beta} F(\bar{w}) + \int_{\bar{w}}^{B} \frac{w}{1-\beta} F(dw)$$

- *Unemployment is voluntary.* Higher UI compensation or higher probability of right-tail offers (with protection from left tail) make the worker choosier, raising $\bar{w}$, unemployment duration, and welfare.

Simple extensions (some you will do)

- same basic reservation wage analysis applies for a firm choosing whether to accept/reject wage bids (period profits falling in w).
- same basic reservation wage analysis applies to the case of multiple offers in a period (costlessly observed)
- analysis also extends to allow nonzero per-period probability of receiving no offer (or allowing worker to choose search intensity)
- allowing recall has no effect on policy or reservation wage
- if quits are allowed, but they incur a 1-period unemployment penalty (permitting no on-the-job-search), option is never exercised
- if worker can be fired in any period with probability $\alpha \in (0,1)$, same policy applies, but with lower $\bar{w}$...

Exogenous firing extension

- With probability $\alpha \in (0,1)$, worker is fired. When fired, worker must sit unemployed for 1 period; thereafter, offers begin.

- Let $\hat{v}^E(w)$ be the value to a worker of being employed with wage w. Value of receiving wage offer w is then:

$$\hat{v}(w) = \max\left\{\hat{v}^E(w),\ c + \beta \int_0^B \hat{v}(w')F(dw')\right\} \tag{A}$$

where

$$\hat{v}^E(w) = w + \beta(1-\alpha)\hat{v}^E(w) + \beta\alpha\left[c + \beta \int_0^B \hat{v}(w')F(dw')\right] \tag{B}$$

- Rearrange (B):

$$\hat{v}^E(w) = \frac{1}{1-\beta(1-\alpha)}\left[w + \beta\alpha\left(c + \beta \int_0^B \hat{v}(w')F(dw')\right)\right] \tag{C}$$

Exogenous firing extension part 2

$$\hat{v}(w) = \max\left\{\hat{v}^E(w),\ c + \beta \int_0^B \hat{v}(w')F(dw')\right\} \tag{A}$$

where

$$\hat{v}^E(w) = \frac{1}{1-\beta(1-\alpha)}\left[w + \beta\alpha\left(c + \beta \int_0^B \hat{v}(w')F(dw')\right)\right] \tag{C}$$

- Put (C) in (A), defining $E\hat{v} \equiv \int_0^B \hat{v}(w)F(dw)$:

$$\hat{v}(w) = \max\left\{\frac{w + \beta\alpha\left[c + \beta E\hat{v}\right]}{1-\beta(1-\alpha)},\ c + \beta E\hat{v}\right\}$$

- Reservation wage solves:

$$\bar{w} + \beta\alpha\left[c + \beta E\hat{v}\right] = [1-\beta(1-\alpha)]c + [1-\beta(1-\alpha)]\beta E\hat{v}$$

Exogenous firing extension part 3

- Reservation wage solves:

$$\bar{w} + \beta\alpha\left[c + \beta E\hat{v}\right] = [1 - \beta(1-\alpha)]c + [1 - \beta(1-\alpha)]\beta E\hat{v}$$

$$\Rightarrow \quad \bar{w} = (1-\beta)\left[c + \beta E\hat{v}\right]$$

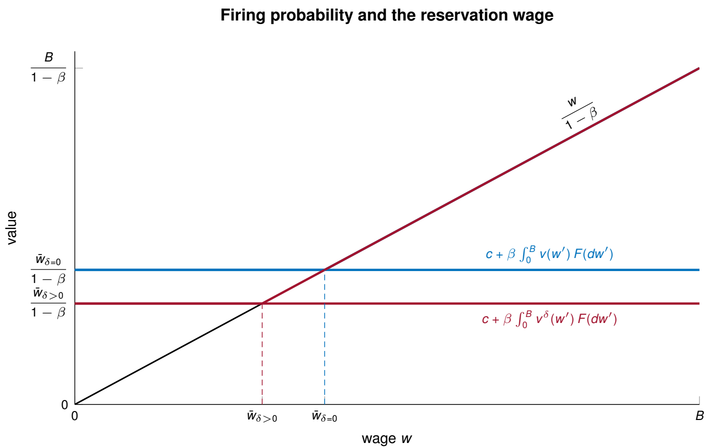
$$\frac{\bar{w}}{1-\beta} = c + \beta \int_0^B \hat{v}(w) F(dw) \tag{D}$$

- Compare (D) to (1). Unaltered reservation wage policy...

- But, since $\hat{v}(w) < v(w)$, the critical $\bar{w}$ is lower with firing.

- Why? *Possibility of being fired lowers expected return to search.*

Firing probability and the reservation wage



- both welfare and the reservation wage fall with firing probability