Graduate Macro Sequence:
Two-Side Labor Search Model

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Two-sided search [Mortensen (1982), Pissarides (1985)]

- Two-sided search: unemployed workers seek jobs and firms with unfilled positions seek workers.
- A CRS matching function transforms measures of unemployment u and vacancies v into new matches $M(u, v)$.
- When a worker and a firm meet, they bargain over the wage w, splitting the **match surplus**.
- Either party can walk away $\Rightarrow$ outside options matter (threat points).
- **Search externality:** agents ignore how their actions affect *labor market tightness* θ and thus other agents' transition rates.
- Except under a knife-edge **Hosios condition**, decentralized equilibrium is **inefficient**.

Preferences and technologies

- Unit measure of identical, infinitely-lived, risk-neutral workers:

$$u(w) = w, \qquad u(c) = c, \qquad \beta \in (0, 1).$$

- Continuum of risk-neutral, profit-maximizing managers (firms), same β (stationary equilibrium).
- Production: $y = zh$, where $h \in \{0, 1\}$ indicates whether the manager is matched with a worker.
- To post (advertise) a vacancy, an unmatched manager pays a per-period cost $p > 0$.
- Free entry by managers implies: expected PDV of a vacancy equals posting cost (zero-profit condition).
- Employment and separations:
    - Let n be the employment rate and $u = 1 - n$ the unemployment rate.
    - Job destruction is exogenous at rate $\delta \in (0, 1)$.
    - Effective discounting for an existing match: $\beta(1 - \delta)$.

More on matching technology

- Matching function $M(u, v)$ is CRS, increasing in u, v, concave.
- CRS $\Rightarrow$ represent as:

$$M(u, v) = u\,M\left(1, \frac{v}{u}\right) = u\,M(\theta), \qquad \theta \equiv \frac{v}{u} \text{ (market tightness)}.$$

- Transition probabilities:

$$q(\theta) \equiv \frac{M(u, v)}{v} = \theta^{-1} M(\theta) \quad \text{(probability a vacancy is filled)},$$

$$f(\theta) \equiv \frac{M(u, v)}{u} = \theta q(\theta) \quad \text{(probability an unemployed worker finds a job)}.$$

- Common CRS concave form: Cobb–Douglas matching

$$M(u, v) = u^{\alpha} v^{1-\alpha} = u\,\theta^{1-\alpha}, \qquad \alpha \in (0, 1).$$

Then

$$q(\theta) = \theta^{-\alpha}, \quad q'(\theta) = -\alpha\theta^{-(\alpha+1)} < 0,$$

and

$$f(\theta) = \theta q(\theta) = \theta^{1-\alpha}, \qquad q(\theta) + \theta q'(\theta) = (1-\alpha)\theta^{-\alpha} > 0.$$

CRS matching technology and Beveridge curve

- CRS evidence (Blanchard & Diamond, 1989): regress $\ln M(u_t, v_t)$ on $\ln u_t$, $\ln v_t$ and test $\alpha + (1-\alpha) = 1$.

- In steady state, job creation equals destruction:

$$M(u, v) = \delta(1-u) \qquad \big(\text{since } n = 1-u\big).$$

- Cobb–Douglas case:

$$u^{\alpha} v^{1-\alpha} - \delta(1-u) = 0.$$

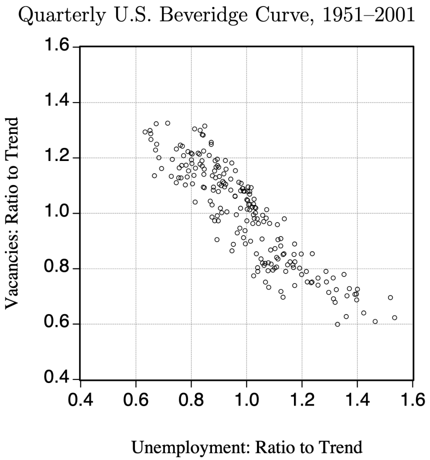
  Totally differentiate (holding parameters fixed) to get a negative slope:

$$\frac{dv}{du} = -\frac{\alpha\theta^{1-\alpha} + \delta}{(1-\alpha)\theta^{-\alpha}} < 0.$$

- General CRS case $M(u, v) = uM(\theta)$ also yields $\frac{dv}{du} < 0$ (a Beveridge curve is downward sloping).

The Beveridge curve

Figure: Shimer 2003 NBER WP



- Beveridge curve: negative relation between unemployment u and vacancies v.
- Model-implied slope (C-D):

$$\frac{dv}{du} = -\frac{\alpha\theta^{1-\alpha} + \delta}{(1-\alpha)\theta^{-\alpha}} < 0.$$

Similar shape as in aggregate data!

Bargaining over match surplus

- When a worker and a manager meet, they bargain over wage w.
- Worker's Nash bargaining power: $\phi \in (0, 1)$.
- Manager's Nash bargaining power: $1 - \phi$.
- If bargaining breaks down, they immediately become unmatched.
- Off-equilibrium threat points:
  - Worker threatens with outside option V_u.
  - Manager threatens with outside option J_u.

Key equations and definitions so far

- Labor market tightness: $\theta \equiv v/u$.

- New matches (hires):

$$M(u, v) = u^{\alpha} v^{1-\alpha} = u\theta^{1-\alpha} \qquad \text{(or } M(u, v) = uM(\theta)\text{)}. \tag{1}$$

- Transition probabilities (manager and worker):

$$q(\theta) \equiv \frac{M(u, v)}{v} = \theta^{-1} M(\theta) \quad \text{(vacancy filling)}. \tag{2}$$

$$f(\theta) \equiv \frac{M(u, v)}{u} = \theta q(\theta) \quad \text{(job finding)}. \tag{3}$$

- Worker's bargaining power: ϕ.

Decentralized equilibrium: necessary conditions

Value of an unemployed worker:

$$V_u = c + \beta\Big(f(\theta)V_e + [1 - f(\theta)]V_u\Big). \tag{4}$$

Value of an employed worker:

$$V_e = w + \beta\Big(\delta V_u + [1 - \delta]V_e\Big). \tag{5}$$

Value of an unmatched manager (posting a vacancy):

$$J_u = -p + \beta\Big(q(\theta)J_e + [1 - q(\theta)]J_u\Big). \tag{6}$$

Value of a matched manager:

$$J_e = z - w + \beta\Big(\delta J_u + [1 - \delta]J_e\Big). \tag{7}$$

Decentralized equilibrium: free entry

Repeat key value equations (workers + managers) and add free entry:

$$V_u = c + \beta\Big(f(\theta)V_e + [1 - f(\theta)]V_u\Big), \tag{4}$$

$$V_e = w + \beta\Big(\delta V_u + [1 - \delta]V_e\Big), \tag{5}$$

$$J_u = -p + \beta\Big(q(\theta)J_e + [1 - q(\theta)]J_u\Big), \tag{6}$$

$$J_e = z - w + \beta\Big(\delta J_u + [1 - \delta]J_e\Big). \tag{7}$$

Free entry of managers:

$$J_u = 0. \tag{8}$$

Plugging $J_u = 0$ into (6) gives the posting-fee relation:

$$p = \beta q(\theta)J_e. \tag{9}$$

Decentralized equilibrium: unemployment evolution

With $u = 1 - n$, unemployment next period satisfies:

$$u' = \delta(1-u) + [1 - f(\theta)]u. \tag{10}$$

Interpretation: next period unemployed are (i) newly separated workers plus (ii) unemployed who fail to find jobs.

Steady state (bathtub condition):

$$\delta(1-u) = uf(\theta) = u\,\theta q(\theta).$$

So the steady-state unemployment rate is

$$u = \frac{\delta}{\delta + \theta q(\theta)}. \tag{11}$$

Nash bargaining problem 1 (pie splitting)

- Match surplus:

$$S = s_v + s_j = (V_e - V_u) + (J_e - J_u).$$

- Worker outside option: V_u. Manager outside option: $J_u = 0$.

Nash bargaining (version 1):

$$\max_{s_v \in [0,S]} \; s_v^{\phi}(S - s_v)^{1-\phi}.$$

FOC implies:

$$\phi(S - s_v) = (1-\phi)s_v \quad \Rightarrow \quad s_v = \phi S, \qquad s_j = (1-\phi)S.$$

Interpretation: bargaining splits total surplus in shares ϕ and $1 - \phi$.

Nash problem and stationary equilibrium defined

Nash bargaining (version 2):

$$\max_{w} \ (V_e - V_u)^{\phi}(J_e - J_u)^{1-\phi} \tag{12}$$

subject to participation constraints:

$$V_e - V_u \geq 0, \qquad J_e - J_u \geq 0. \tag{13}$$

The same pie-sharing condition can be written as:

$$(V_e - V_u) = \phi S, \qquad (J_e - J_u) = (1-\phi)S, \qquad S = (V_e - V_u) + (J_e - J_u).$$

Steady-state equilibrium objects:

$$\{V_u, V_e, J_u, J_e, \theta, u, w\}$$

satisfying (4),(5),(7),(8),(9),(11) and the Nash FOC.

Determining worker match surplus

From worker value equation (5):

$$V_e = w + \beta\Big(\delta V_u + (1-\delta)V_e\Big).$$

Rearrange in terms of the worker surplus $V_e - V_u$:

$$V_e - V_u = w - (1-\beta)V_u + \beta(1-\delta)(V_e - V_u).$$

Thus

$$V_e - V_u = \frac{w - (1-\beta)V_u}{1-\beta(1-\delta)}. \tag{14}$$

Determining manager match surplus

With free entry $J_u = 0$, the matched manager value (7) becomes:

$$J_e = z - w + \beta(1-\delta)J_e \quad \Rightarrow \quad J_e = \frac{z - w}{1 - \beta(1-\delta)}. \tag{16}$$

- (16) implies $w < z$ is required for $J_e \geq 0$.
- V_u does not depend on w directly, but it depends on θ, which depends on equilibrium.
- Sensitivity of worker surplus to w:

$$\frac{\partial(V_e - V_u)}{\partial w} = \frac{1}{1 - \beta(1-\delta)}.$$

Interior solution to Nash bargaining

Nash problem:

$$\max_{w} \; (V_e - V_u)^{\phi}(J_e)^{1-\phi}. \tag{12}$$

FOC (using $\partial(V_e - V_u)/\partial w = 1/[1-\beta(1-\delta)]$ and $\partial J_e/\partial w = -1/[1-\beta(1-\delta)]$):

$$\phi \frac{1}{V_e - V_u} \frac{1}{1-\beta(1-\delta)} \; + \; (1-\phi)\frac{1}{J_e}\left(-\frac{1}{1-\beta(1-\delta)}\right) = 0.$$

This simplifies to the key surplus-sharing condition:

$$(1-\phi)(V_e - V_u) = \phi J_e. \tag{17}$$

Next: combine (17) with (14) and (16) to derive the wage.

Equilibrium wage splits per-period surplus

Use (17): $(1-\phi)(V_e - V_u) = \phi J_e$, together with (14) and (16):

$$V_e - V_u = \frac{w - (1-\beta)V_u}{1-\beta(1-\delta)}, \qquad J_e = \frac{z - w}{1-\beta(1-\delta)}.$$

Multiply by $1-\beta(1-\delta)$ and rearrange:

$$(1-\phi)\left[w - (1-\beta)V_u\right] = \phi(z - w).$$

Solve for wage:

$$w = \phi z + (1-\phi)(1-\beta)V_u. \tag{18}$$

Interpretation: wage equals worker's outside option plus a ϕ-share of match surplus.

Wage as function of labor market tightness

Start from wage (18):

$$w = \phi z + (1-\phi)(1-\beta)V_u.$$

From unemployed worker value (4):

$$(1-\beta)V_u = c + \beta f(\theta)(V_e - V_u).$$

From Nash condition (17): $(1-\phi)(V_e - V_u) = \phi J_e$. Using free entry $p = \beta q(\theta) J_e$ (9), we get:

$$(1-\beta)V_u = c + \beta f(\theta)\frac{\phi}{1-\phi}J_e = c + \frac{\phi}{1-\phi}\theta p. \quad (\dagger)$$

Plug (†) into (18):

$$w = \phi z + (1-\phi)c + \phi\theta p. \tag{19}$$

This is the **first key** $w(\theta)$ equation.

Wage comparative statics

From

$$w = \phi z + (1-\phi)c + \phi\theta p, \tag{19}$$

- $\partial w/\partial z > 0$ (productivity raises wages)
- $\partial w/\partial c > 0$ (better outside option raises wages)
- $\partial w/\partial \theta > 0$ (tight markets raise wages)
- $\partial w/\partial \phi > 0$ (more worker power raises wages)
- Vacancy cost p: direct effect raises $\phi\theta p$, but equilibrium θ may fall when p rises.

$$dw = \phi(\theta\, dp + p\, d\theta) \quad \Rightarrow \quad \frac{dw}{dp} = \phi\theta + \phi p \frac{d\theta}{dp}.$$

So the sign of dw/dp depends on how strongly θ falls with p.

Closing the model: solving market tightness

We have two key equilibrium relationships involving θ:

- From wage equation (19):
$$z - w = (1-\phi)(z-c) - \phi\theta p.$$
- From matched manager value (16) and free entry (9):
$$J_e = \frac{z-w}{1-\beta(1-\delta)}, \qquad p = \beta q(\theta) J_e \Rightarrow (z-w)\beta q(\theta) = p\left[1-\beta(1-\delta)\right].$$

Substitute $z - w$ from the first into the second and rearrange to get an implicit equation for θ:

$$z - c = \frac{1-\beta[(1-\delta)-\phi\theta q(\theta)]}{(1-\phi)\beta q(\theta)}\, p. \tag{20}$$

This θ ensures zero expected profits for vacancies and Nash bargaining shares.

Market tightness in decentralized equilibrium

(A) Tightness is pinned down by (20):

$$(z-c)p^{-1} = \frac{1-\beta[(1-\delta)-\phi\theta q(\theta)]}{(1-\phi)\beta q(\theta)}.$$

(B) Matching technology properties (CRS, increasing, concave):

$$q(\theta) \text{ falls in } \theta, \qquad \theta q(\theta) \text{ rises in } \theta.$$

(C) Wage rises in θ: from (19),

$$w = \phi z + (1-\phi)c + \phi\theta p.$$

Comparative statics intuition via (20):

- $p \uparrow \Rightarrow \theta \downarrow$ (vacancies are more expensive) $\Rightarrow$ tends to reduce w.
- $z \uparrow \Rightarrow \theta \uparrow \Rightarrow$ wages rise.
- $c \uparrow \Rightarrow \theta \downarrow$ (workers are pickier / surplus smaller) $\Rightarrow$ wages may be pulled down via θ, but c directly raises wages in (19).

Planner's problem with search frictions

Planner removes strategic behavior (bargaining) and chooses allocation to maximize discounted output plus home production minus vacancy costs, subject to matching.

No uncertainty; aggregates match individual transition probabilities. Employment evolves as:

$$n_{t+1} = (1-\delta)n_t + v_t q(\theta_t), \qquad \theta_t \equiv \frac{v_t}{1-n_t}.$$

Sequence problem:

$$\max_{\{v_t, n_{t+1}\}_{t\geq 0}} \sum_{t=0}^{\infty} \beta^t \left[z n_t + c(1-n_t) - p v_t \right] \tag{22}$$

s.t.

$$n_{t+1} \leq (1-\delta)n_t + v_t q\left(\frac{v_t}{1-n_t}\right). \tag{23}$$

Recursive formulation of planner's problem

Let $W(n)$ be planner value when employment is n. Bellman form:

$$W(n) = \max_{v,n'} \left\{ zn + c(1-n) - pv + \beta W(n') + \Omega\left[(1-\delta)n + vq\left(\tfrac{v}{1-n}\right) - n'\right] \right\}.$$

FOC w.r.t. v:

$$-p + \Omega\Big[q(\theta) + \theta q'(\theta)\Big] = 0 \quad \Rightarrow \quad p = \Omega\Big[q(\theta) + \theta q'(\theta)\Big]. \tag{24}$$

FOC w.r.t. n':

$$\Omega = \beta W'(n'). \tag{25}$$

Benveniste–Scheinkman:

$$W'(n) = z - c + \Omega\Big[1 - \delta + \theta^2 q'(\theta)\Big]. \tag{26}$$

Planner's solution

Collect key planner equations:

$$p = \Omega\big[q(\theta) + \theta q'(\theta)\big], \qquad \Omega = \beta W'(n'), \qquad W'(n) = z - c + \Omega\big[1 - \delta + \theta^2 q'(\theta)\big].$$

In steady state, n, θ, Ω are constant, so combining (25) and (26) gives:

$$z - c = \Omega \frac{1 - \beta[1 - \delta + \theta^2 q'(\theta)]}{\beta}. \tag{27}$$

Combine (24) and (27) to get the planner's implicit condition for θ:

$$z - c = \frac{p}{q(\theta) + \theta q'(\theta)} \cdot \frac{1 - \beta[1 - \delta + \theta^2 q'(\theta)]}{\beta}. \tag{28}$$

Comparison: equilibrium versus planner's solution

Decentralized equilibrium θ solves (20):

$$z - c = \frac{1 - \beta[(1-\delta) - \phi\theta q(\theta)]}{(1-\phi)\beta q(\theta)}\, p. \tag{20}$$

Planner θ solves (28):

$$z - c = \frac{p}{q(\theta) + \theta q'(\theta)} \cdot \frac{1 - \beta[1 - \delta + \theta^2 q'(\theta)]}{\beta}. \tag{28}$$

The two coincide (efficiency) when the **Hosios condition** holds:

$$\phi = -\frac{\theta q'(\theta)}{q(\theta)}.$$

(Equivalently, worker bargaining power equals the elasticity of matches w.r.t. unemployment.)

Interpreting the Hosios condition

Define the elasticity of matches w.r.t. unemployment:

$$\eta_{M,u} \equiv \frac{\partial M(u,v)}{\partial u} \cdot \frac{u}{M(u,v)}.$$

With CRS form $M(u,v) = uM(\theta)$, $\theta = v/u$:

$$\frac{\partial M}{\partial u} = M(\theta) + uM'(\theta)\left(-\frac{v}{u^2}\right) = M(\theta) - \theta M'(\theta).$$

Hence

$$\eta_{M,u} = \frac{M(\theta) - \theta M'(\theta)}{M(\theta)}.$$

Using $q(\theta) \equiv M(u,v)/v = \theta^{-1}M(\theta) \Rightarrow M(\theta) = \theta q(\theta)$, and differentiating $q(\theta) = \theta^{-1}M(\theta)$ yields

$$q'(\theta) = \frac{1}{\theta}\left(M'(\theta) - \frac{M(\theta)}{\theta}\right) \quad \Rightarrow \quad M'(\theta) - \frac{M(\theta)}{\theta} = \theta q'(\theta).$$

Substitute into $\eta_{M,u}$:

$$\eta_{M,u} = -\frac{\theta q'(\theta)}{q(\theta)}.$$

Thus Hosios can be written as $\phi = \eta_{M,u}$.

Planner's solution with C-D matching function

Impose Cobb–Douglas matching, so $q(\theta) = \theta^{-\alpha}$ and $q'(\theta) = -\alpha\theta^{-(\alpha+1)}$.
Planner condition (28):

$$z - c = \frac{p}{q(\theta) + \theta q'(\theta)} \cdot \frac{1 - \beta[1 - \delta + \theta^2 q'(\theta)]}{\beta}.$$

Since $q(\theta) + \theta q'(\theta) = (1-\alpha)\theta^{-\alpha}$ and $\theta^2 q'(\theta) = -\alpha\theta^{1-\alpha} = -\alpha\theta q(\theta)$,

$$z - c = \frac{1 - \beta[1 - \delta - \alpha\theta q(\theta)]}{(1-\alpha)\beta q(\theta)}\, p. \tag{29}$$

Recall decentralized condition (20):

$$z - c = \frac{1 - \beta[(1 - \delta) - \phi\theta q(\theta)]}{(1-\phi)\beta q(\theta)}\, p. \tag{20}$$

Planner's solution versus decentralized outcome

Planner chooses θ via:

$$(z-c)p^{-1} = \frac{1-\beta[1-\delta-\alpha\theta q(\theta)]}{(1-\alpha)\beta q(\theta)}.$$

Decentralized equilibrium chooses θ via:

$$(z-c)p^{-1} = \frac{1-\beta[(1-\delta)-\phi\theta q(\theta)]}{(1-\phi)\beta q(\theta)}.$$

Same outcome only if Hosios condition holds:

$$\phi = \alpha \qquad (\text{equivalently } \phi = \eta_{M,u}).$$

Interpretation (externality): firms do not internalize how posting a vacancy changes θ and thus other agents' matching probabilities.

Direction:

- If $\phi < \alpha$: too many vacancies posted (θ too high).
- If $\phi > \alpha$: too few vacancies posted (θ too low).

Alternative interpretation of Hosios condition

Planner implements efficiency by choosing bargaining weights so that private incentives replicate the planner's θ.

When ϕ rises, the RHS of the decentralized θ-condition increases, so θ must fall to restore equality.

Note (elasticity): elasticity of vacancy filling rate w.r.t. vacancies (C-D):

$$\eta_{q,v} = \left(\frac{\partial q(\theta)}{\partial \theta} \cdot \frac{\partial \theta}{\partial v}\right) \frac{v}{q(\theta)} = \left(-\alpha\theta^{-\alpha-1} \cdot \frac{1}{u}\right) \frac{v}{\theta^{-\alpha}} = -\alpha.$$

If α is large, one extra vacancy has a large negative effect on vacancy-filling rates for all firms. Planner response: curtail vacancies by allocating less bargaining power to workers (so that private vacancy posting is reduced).

Search in an RBC model (preview)

- Search frictions provide an additional propagation mechanism for business cycles.
- Merz (1995, JME) embeds DMP search into an RBC framework:
  - aggregate productivity shocks move z,
  - tightness θ and unemployment u respond endogenously,
  - wages and vacancies co-move with output in ways RBC without search cannot match.
- This is the bridge from “static labor supply” to “frictional labor market” RBC.