RBC Topic 1A: Business-Cycle Facts and the HP Filter

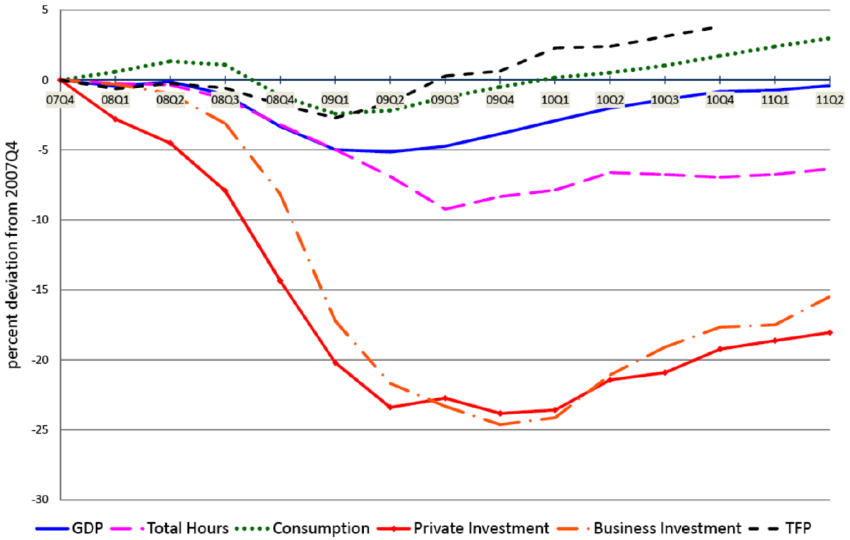
Hui-Jun Chen

National Tsing Hua University
Department of Economics

April 21, 2026

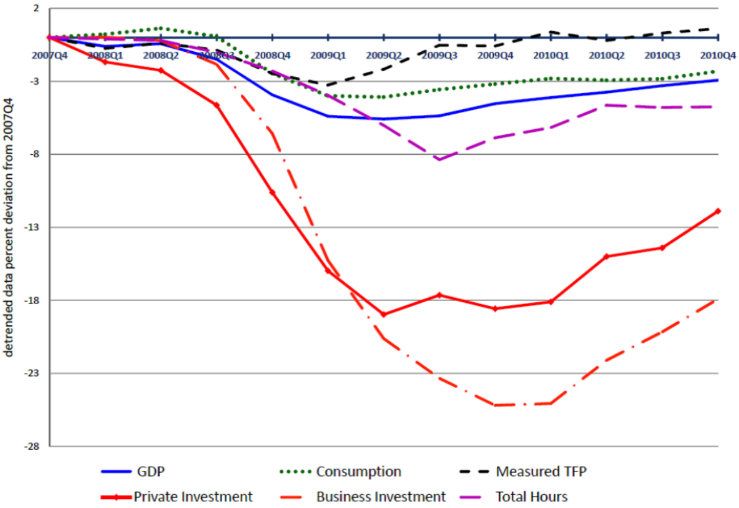
Business-Cycle Facts

The recent U.S. recession: unfiltered first look



The recent U.S. recession: HP-filtered data

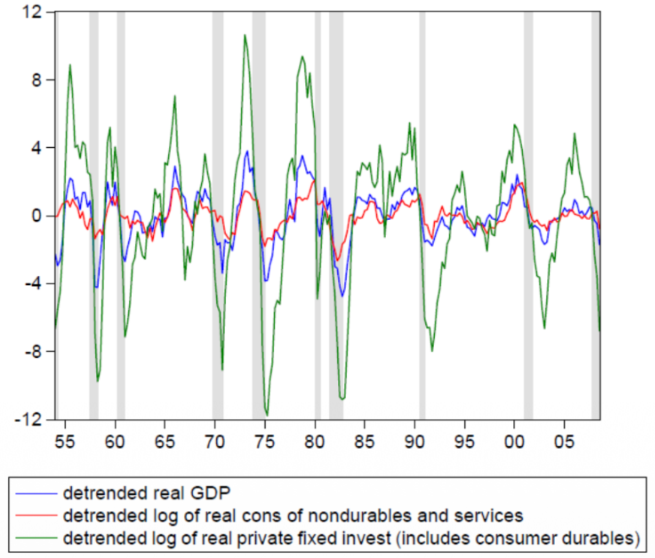
FIGURE 8. The Recent U.S. Recession



Basic business cycle facts (stylized)

time series	relative S. D.	corr.	share of GDP
GDP	1.62	1.00	
consumption	0.50	0.76	0.564
investment	2.78	0.90	0.239
equipment	3.08	0.83	0.072
nonresidential structures	3.55	0.40	0.036
residential structures	6.30	0.65	0.047
consumer durables	2.83	0.78	0.084
government	1.06	-0.02	0.205
net exports	0.24	-0.35	
exports	3.59	0.21	0.080
imports	3.10	0.60	0.092
total hours	0.81	0.83	
final sales	0.77	0.94	0.995
net inventory invest.	0.35	0.60	0.006

Basic business cycle facts: detrended GDP, consumption, investment



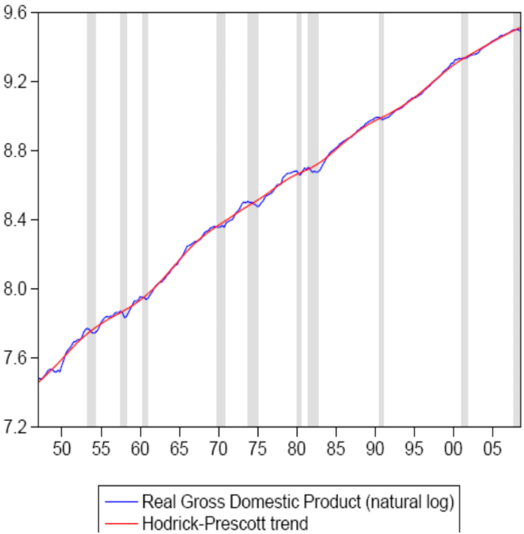
Where we are going in RBC

1. Detrend the data: Hodrick–Prescott (HP) filter
2. Growth in the model: exogenous technical progress
   2.1 Characterize the balanced growth path (BGP)
   2.2 Growth-deflate to study stationary series
3. Calibrate steady state to long-run data (preview)
4. Solve the growth-deflated model (we postpone linearization / KPR methods)

Goal for today: Learn how we measure "cycles" in data (HP filter), then build a model that can speak to these facts.

HP Filter

Postwar U.S. GDP and its HP trend



HP filter: the minimization problem

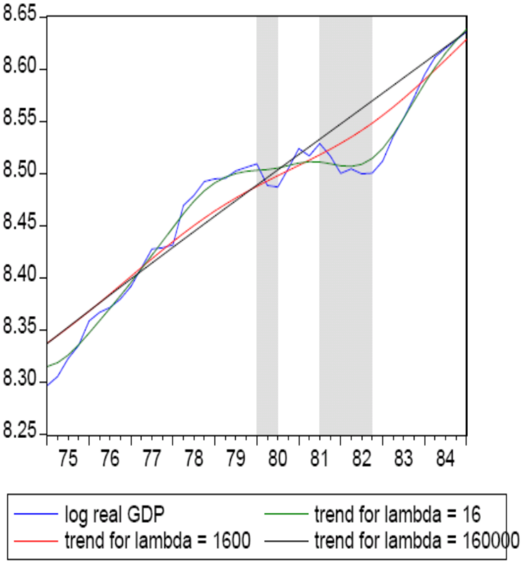
Given data $\{y_t\}_{t=1}^{T}$, the HP filter chooses a trend $\{\tau_t\}_{t=1}^{T}$ by

$$\min_{\{\tau_t\}_{t=1}^{T}} \frac{1}{T}\sum_{t=1}^{T}(y_t - \tau_t)^2 + \frac{\lambda}{T-1}\sum_{t=2}^{T-1}\Big[(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\Big]^2, \qquad \text{(HP)}$$

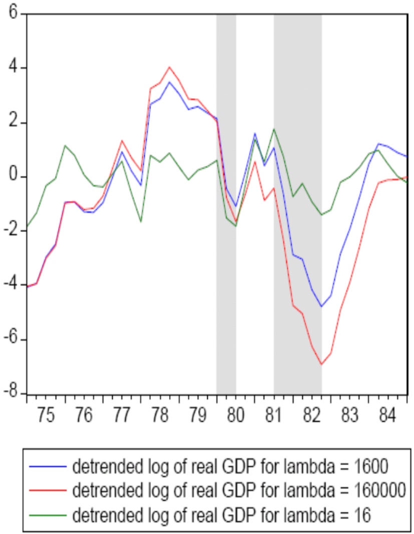
where $\lambda > 0$ controls smoothness.

- First term: fit (trend close to data).
- Second term: smoothness (penalize changes in trend growth; a second-difference penalty).
- Detrended (cycle) series: $\tilde{y}_t \equiv y_t - \tau_t$.

How λ affects the identified HP trend



How λ affects identified business cycles



HP objective expanded: see where FOCs come from

The objective is a quadratic function of $(\tau_1, \ldots, \tau_T)$:

$$\sum_{t=1}^{T} (y_t - \tau_t)^2 \; + \; \lambda \sum_{t=2}^{T-1} \big(\tau_{t+1} - 2\tau_t + \tau_{t-1}\big)^2.$$

The penalty term uses the second difference $\Delta^2 \tau_t \equiv \tau_{t+1} - 2\tau_t + \tau_{t-1}$.

Generic interior FOC idea: for $2 < t < T-1$, τ_t appears in $(y_t - \tau_t)^2$ and in three nearby penalty terms: $\Delta^2\tau_{t-1}$, $\Delta^2\tau_t$, $\Delta^2\tau_{t+1}$.

The ordinary first-order condition (interior dates)

For an interior date t with $2 < t < T - 1$, the FOC is a linear difference equation involving $\tau_{t-2}, \tau_{t-1}, \tau_t, \tau_{t+1}, \tau_{t+2}$:

$$2(y_t - \tau_t) + 2\lambda\Big[(\tau_t - \tau_{t-1}) - (\tau_{t-1} - \tau_{t-2}) - 2\big((\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})\big) + (\tau_{t+2} - \tau_{t+1}) - (\tau_{t+1} - \tau_t)\Big] = 0.$$

- This is a fifth-order linear difference equation (uses τ_{t-2} through τ_{t+2}).
- Endpoint dates have “missing” terms and hence different FOCs.

Endpoint terms: why first and last dates differ

At $t = 1, 2, T-1, T$, some second-difference terms do not exist, so the FOCs change.

- Example intuition: τ_1 only shows up in $(y_1 - \tau_1)^2$ and $\Delta^2\tau_2$ (not in $\Delta^2\tau_0$).
- Similarly, τ_T only shows up in $(y_T - \tau_T)^2$ and $\Delta^2\tau_{T-1}$.

Takeaway: HP filtering is a global smoothing problem; endpoints are special.

HP filter as a linear system: $y = A\tau$

Because the objective is quadratic, the necessary and sufficient conditions are a linear system:

$$y = A\tau, \quad y \equiv [y_1, \dots, y_T]^\top, \quad \tau \equiv [\tau_1, \dots, \tau_T]^\top.$$

The matrix A is symmetric banded (nonzero elements on 5 diagonals). Its interior pattern is:

$$A_{t,t} = 6\lambda + 1, \quad A_{t,t\pm 1} = -4\lambda, \quad A_{t,t\pm 2} = \lambda, \qquad (2 < t < T - 1),$$

with modified coefficients near the endpoints.

Compute: $\tau = A^{-1}y$, then detrend by $\tilde{y} = y - \tau$.